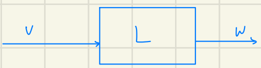
Linear system is a function that satisfies the principle of superposition (linearity)



$L(v_1+v_2) = Lv_1 + Lv_2$

$L(\alpha v) = \alpha L v$

$\left(L\left(\sum_i \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i L v_i \right)$

can extend these ideas to infinite sum (and integrals) under some assumptions on the operator L

eg. direct proportionality

$Lv = \alpha v$ all linear systems harken back to direct proportionality (multiplication)

eg. $Lv(t) = \alpha(t)\cdot v(t)$

eg. switch $(Lv)(t) = \Pi_a(t)\cdot v(t)$

eg. sampling $(Lv)(t) = \text{Ш}_p(t)\cdot v(t)$

slight but important generalization: direct proportionality + adding

eg matrix multiplication

$A \in \mathbb{R}^{n\times m}$ $v \in \mathbb{R}^m$ $A(v+w) = Av + Aw$ $A(\alpha v) = \alpha A v$

special linear systems with special properties derive from special property of the matrix A

symmetry $A^T = A$, hermitian $A^* = \bar{A}^T = A$, orthogonal $AA^T = I$

important to understand eigenvalues and eigenfunctions associated with A

$Av = \lambda v$ $v \neq 0$ direct proportionality

$Ax = A\sum_i \alpha_i x_i = \sum_i \alpha_i \lambda_i x_i$ direct proportionality + adding

(hermitian / symmetry $\rightarrow$ orthogonal basis of eigenvectors)

any finite-dimensional linear system can be realized as matrix multiplication

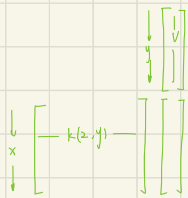
eg. input $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$L = \frac{d}{dx}$ which is a linear operator

$$\begin{array}{cc} & n+1 \\ n & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & 0 & n \end{bmatrix} \end{array} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \end{bmatrix}$$

analogous statement for infinite continuous case

matrix multiplication $\xrightarrow{\text{analogy}}$ integration against a kernel



input: function $v(x)$

kernel: function $k(x,y)$

$Lv(x) = \int_{-\infty}^{+\infty} k(x,y)\,v(y)\,dy$

L is a linear system

special properties of A $\xrightarrow{\text{analogy}}$ special properties of kernel

symmetry: $A_{ji} = A_{ij}$ $\quad$ $k(x,y) = k(y,x)$ (self-adjoint)

hermitian $A_{ji} = \overline{A_{ij}}$ $\quad$ $k(x,y) = \overline{k(y,x)}$

linear system given by integration against the kernel

eg. The F.T. $\mathcal{F}f(s) = \int_{-\infty}^{+\infty} f(t)\, e^{-2\pi i s t}\, dt$ $\quad$ $k(s,t) = e^{-2\pi i s t}$

eg. convolution

for a function h, the operator $Lv = h * v$

$(Lv)(x) = \int_{-\infty}^{+\infty} h(x-y)\, v(y)\, dy$ $\quad$ the kernel $k(x,y) = h(x-y)$ only depends on the difference

(circulant A) shift-invariance / time-invariance

$x \to x-a$ $\quad$ $y \to y-a$ $\quad$ $(x-y) \to (x-y)$

any linear system can be written as integration against a kernel