$$\begin{array}{c} \longrightarrow n \longleftarrow \\ n \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-2\pi i \frac{1\cdot 1}{N}} & & e^{-2\pi i \frac{(N-1)\cdot 1}{N}} \\ 1 & e^{-2\pi i \frac{1\cdot 2}{N}} & & \vdots \\ \vdots & \vdots & & \\ 1 & e^{-2\pi i \frac{1(N-1)}{N}} & & e^{-2\pi i \frac{(N-1)(N-1)}{N}} \end{bmatrix} \begin{bmatrix} \\ f \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \mathcal{F}f \\ \\ \end{bmatrix} \\ W^{-0} \qquad W^{-1} \qquad\qquad W^{-(N-1)} \end{array} \qquad O(n^2)$$

(1 approach to FFT is to write the DFT matrix as the product of sparse matracies)

$$\mathcal{F}f[m] = \sum_{n=0}^{N-1} f[n]\, e^{-2\pi i \frac{nk}{N}}$$

write the sum as sum over odd indices and over even indicies

that is write the n-order DFT as 2 $\frac{n}{2}$-order DFTs

(Assume n is a power of 2)

use notation $W[p,q] = e^{2\pi i\, q/p}$ $\qquad W[p, q_1+q_2] = W[p,q_1]\cdot W[p,q_2]$

$W[\frac{N}{2}, -mn] = W[N, -2mn]$ $\qquad$ A power (mn) of half-order $(\frac{N}{2})$ exponential is an even power $(2mn)$ of full order (N) exponential

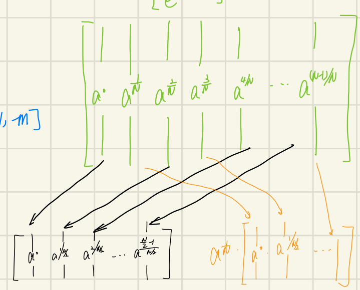
$$\begin{aligned} W[N, -(2n+1)m] &= W[N, -2mn - m] \\ &= W[N, -2mn]\cdot W[N,-m] \\ &= W[N, -mn]\cdot W[N,-m] \end{aligned}$$

even power: $W[N, -2mn] = W[\frac{N}{2}, -mn]$

odd power: $W[N, -(2n+1)m] = W[N, -mn]\cdot W[N,-m]$

$$\begin{aligned} \mathcal{F}f[m] &= \sum_{n=0}^{N-1} f[n]\cdot e^{-2\pi i \frac{mn}{N}} \\ &= \sum_{n=0}^{N-1} f[n]\cdot W[N,-mn] \\ &= \sum_{n=0}^{\frac{N}{2}-1} f[2n]\cdot W[N,-2mn] + \sum_{n=0}^{\frac{N}{2}-1} f[2n+1]\cdot W[N, -m(2n+1)] \\ &= \sum_{n=0}^{\frac{N}{2}-1} f[2n]\cdot W[\tfrac{N}{2}, -mn] + \sum_{n=0}^{\frac{N}{2}-1} f[2n+1]\cdot W[N,-2mn]\cdot W[N,-m] \\ &= \sum_{n=0}^{N/2-1} f[2n]\cdot W[\tfrac{N}{2}, -mn] + W[N,-m]\sum_{n=0}^{\frac{N}{2}-1} f[2n+1]\cdot W[N,-2mn] \\ &= \sum_{n=0}^{N/2-1} f[2n]\cdot W[\tfrac{N}{2}, -mn] + W[N,-m]\sum_{n=0}^{N/2-1} f[2n+1]\cdot W[\tfrac{N}{2}, -mn] \end{aligned}$$

let $a = \begin{bmatrix} e^{-2\pi i\cdot 0} \\ e^{-2\pi i\cdot 1} \\ e^{-2\pi i\cdot 2} \\ \vdots \\ e^{-2\pi i\cdot (N-1)} \end{bmatrix}$



for $0 \le m \le \frac{N}{2}-1$, $\qquad \mathcal{F}_N f[m] = \mathcal{F}_{N/2} f_{even}[m] + e^{-2\pi i \frac{m}{N}} \mathcal{F}_{N/2} f_{odd}[m]$

for $[\frac{N}{2}, N-1]$, write as

$$\mathcal{F}f[m+\tfrac{N}{2}] = \sum_{n=0}^{N/2-1} f[2n]\cdot W[\tfrac{N}{2}, -(\tfrac{N}{2}+m)n] + W[N, -(m+\tfrac{N}{2})] \sum_{n=0}^{N/2} f[2n+1]\cdot W[\tfrac{N}{2}, -(m+\tfrac{N}{2})n]$$

$$\begin{bmatrix} a^0 & a^{\frac{1}{N/2}} & \cdots & a^{\frac{N/2-1}{N/2}} \end{bmatrix} = \begin{bmatrix} e^{-2\pi i \frac{0}{N/2}} & \\ \vdots & \\ e^{-2\pi i \frac{N/2-1}{N/2}} & \cdots \\ e^{-2\pi i \frac{N/2}{N/2}} & \\ \vdots & \\ e^{-2\pi i \frac{N-1}{N/2}} & \end{bmatrix} \; N$$

$N/2$

periodicity

$$W[\tfrac{N}{2}, -(\tfrac{N}{2}+m)n]$$
$$= W[\tfrac{N}{2}, -\tfrac{Nn}{2}]\cdot W[\tfrac{N}{2}, -mn]$$
$$= e^{-2\pi i \frac{-Nn/2}{N/2}} \cdot W[\tfrac{N}{2}, -mn]$$
$$= 1 \cdot W[\tfrac{N}{2}, -mn]$$

periodicity

$$= \sum_{n=0}^{N/2-1} f[2n]\cdot W[\tfrac{N}{2}, -mn] + W[N,-m]\, W[N, -\tfrac{N}{2}] \sum_{n=0}^{N/2} f[2n+1]\cdot W[\tfrac{N}{2}, -mn]$$

$$W[N, -N/2] = e^{-2\pi i \frac{N/2}{N}} = e^{-\pi i} = -1$$

$$= \sum_{n=0}^{N/2-1} f[2n]\cdot W[\tfrac{N}{2}, -mn] - W[N,-m] \sum_{n=0}^{N/2} f[2n+1]\cdot W[\tfrac{N}{2}, -mn]$$

$$\mathcal{F}_N f = \begin{bmatrix} \begin{bmatrix} \mathcal{F}_{N/2} f_{even} \end{bmatrix} + \begin{bmatrix} e^{-2\pi i \frac{k}{m}} \end{bmatrix} \begin{bmatrix} \mathcal{F}_{N/2} f_{odd} \end{bmatrix} \\ \begin{bmatrix} \mathcal{F}_{N/2} f_{even} \end{bmatrix} - \begin{bmatrix} e^{-2\pi i \frac{k}{m}} \end{bmatrix} \begin{bmatrix} \mathcal{F}_{N/2} f_{odd} \end{bmatrix} \end{bmatrix}$$

essence of FFT

$$\begin{bmatrix} e^{-2\pi i \frac{k\cdot 0}{N}} \\ e^{-2\pi i \frac{k\cdot 1}{N}} \\ e^{-2\pi i \frac{k\cdot 2}{N}} \\ \vdots \\ e^{-2\pi i \frac{k(N-1)}{N}} \end{bmatrix}$$

w^k

$$e^{-2\pi i \frac{k\cdot(m+\frac{N}{2})}{N}} = e^{-2\pi i \frac{km}{N}} \cdot e^{-k\pi i}$$

for even k, has period $\frac{N}{2}$

for odd k, has period $\frac{N}{2}$ with a flip of sign

$$\begin{bmatrix} + & + & + & + & \\ & & & & \cdots \\ + & - & + & - & \end{bmatrix}$$

FFT does no more than "only calculating the 'plus' term"