$$\mathcal{F}f = \sum_{k=0}^{N-1} f[k]\, w^{k}$$

$$\mathcal{F}^{-1}f = \frac{1}{N} \sum_{k=0}^{N-1} f[k]\, w^{k}$$

$$w = \left(1,\ e^{2\pi i \frac{1}{N}},\ \cdots,\ e^{2\pi i \frac{N-1}{N}}\right)$$

· Special cases

Discrete	Continuous
$\mathcal{F}f[0] = \sum_{k=0}^{N-1} f[k] \cdot e^{2\pi i \frac{0k}{N}} = \sum_{k=0}^{N-1} f[k]$	$\mathcal{F}f(0) = \int_{-\infty}^{+\infty} f(t)\, dt$

delta: $\delta_0 = [1, 0, 0, \cdots]$

$$(\mathcal{F}\delta)(s) = \sum_{k=0}^{N-1} \delta[k]\, e^{-2\pi i \frac{sk}{N}} = 1 \qquad\qquad (\mathcal{F}\delta)(s) = 1$$

$$\left.\begin{aligned} (\mathcal{F}\delta_p)(s) &= \sum_{k=0}^{N-1} \delta[k]\, e^{-2\pi i \frac{sk}{N}} = e^{-2\pi i ps/N} \\ \mathcal{F}\delta_p &= w^{-p} \end{aligned}\right\} \qquad (\mathcal{F}\delta_p)(s) = e^{-2\pi i ps}$$

exponential:

$$\left.\begin{aligned} (\mathcal{F}w^p)(s) &= \sum_{k=0}^{N-1} e^{2\pi i \frac{kp}{N}} \cdot e^{-2\pi i \frac{ks}{N}} \\ &= \langle w^p, w^s \rangle = \begin{cases} 0 & p \neq s \\ N & p = s \end{cases} \\ (\mathcal{F}w^p)(s) &= N \cdot \delta_p \end{aligned}\right\} \qquad \mathcal{F}\left(e^{2\pi i p t}\right)(s) = \delta_p$$

· Periodicity

must consider the input and output of DFT as periodic with period N

the complex exponential is itself periodic

$$w^k[m] = e^{2\pi i \frac{mk}{N}} = e^{2\pi i \frac{k(m+N)}{N}} = w^k[m+N]$$

when forming DFT, we're forming linear combinations of periodic signals of period N

$\mathcal{F}f$ has period N

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\begin{array}{c} \updownarrow N \\ \updownarrow N \end{array} \begin{bmatrix} w & w \\ w & w \end{bmatrix} \begin{bmatrix} \mathcal{F}f_1 + \mathcal{F}f_2 \\ \mathcal{F}f_1 + \mathcal{F}f_2 \end{bmatrix} \to \text{period } N$$

$$\begin{array}{c} \updownarrow N \\ \updownarrow N \end{array} \begin{bmatrix} \overline{w} & \overline{w} \\ \overline{w} & \overline{w} \end{bmatrix} \begin{bmatrix} f_1 + f_2 \\ f_1 + f_2 \end{bmatrix} \to \text{period } N$$

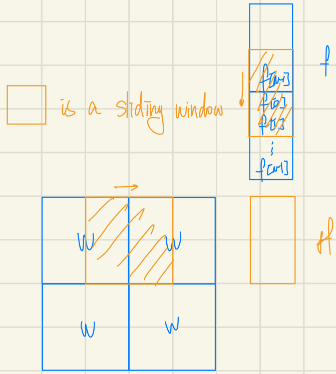
$\to$ consider input to DFT as periodic with period N

Simple but helpful consequence of periodicity : independence of indexing

$$\mathcal{F}f = \sum_{k=0}^{N-1} f[k]\, \omega^{-k}$$

$$= \sum_{k=1}^{N} f[k]\, \omega^{-k}$$

$$= \begin{cases} \displaystyle\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} f[k]\, \omega^{-k} & N \text{ even} \\ \displaystyle\sum_{k=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} f[k]\, \omega^{-k} & N \text{ odd} \end{cases}$$



· Duality

$$f^{-}[n] = f[-n]$$

$$\mathcal{F}(f^{-}) = (\mathcal{F}f)^{-} \qquad \mathcal{F}\mathcal{F}f = N f^{-}$$

$$f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix} \qquad f^{-} = \begin{bmatrix} f^{-}_0 = f[0] = f[0] \\ f^{-}[1] = f[-1] = f[N-1] \\ \vdots \\ f^{-}[N-1] = f[1-N] = f[1] \end{bmatrix}$$

$$f^{-} = Pf = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \ddots \end{array}\right] f$$

$$W = \begin{bmatrix} | & | & & | \\ \omega^0 & \omega^1 & \cdots & \omega^{N-1} \\ | & | & & | \end{bmatrix}$$

$$WP = \begin{bmatrix} \omega^0 & \omega^{N-1} & \cdots & \omega^2 & \omega^1 \end{bmatrix} = \begin{bmatrix} \omega^0 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-N} \end{bmatrix} = \bar{W} \qquad \bar{W}P = WPP = W$$

$$PW = (W^T P^T)^T = (WP)^T = \bar{W}$$

$$P\bar{W} = PPW = W$$

1° $\mathcal{F}(f^{-}) = WPf = PPWPf = P\bar{W}Pf = PWf = (\mathcal{F}f)^{-}$

$$\mathcal{F}(f^{-}) = (\mathcal{F}f)^{-}$$

2° $\mathcal{F}\mathcal{F}f = WWf = WPPWf = WP(\mathcal{F}f)^{-} = WP\mathcal{F}(f^{-}) = WPWPf = \bar{W}Wf^{-} = Nf^{-}$

$$\mathcal{F}\mathcal{F}f = Nf^{-}$$