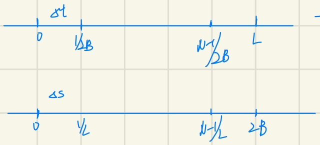
Final definition of DFT $\quad F[m] = \sum_{k=0}^{N-1} f[k]\, e^{-2\pi i mk/N}$



$\Delta t \cdot \Delta s = \frac{2BL}{N^2} = \frac{1}{N}$ } reciprocal relationship between spacings in time and frequency domain

Δs is determined by Δt and N

$f = (f[0], f[1], \cdots, f[N-1])$

$\mathcal{F}f[m] = \sum_{k=0}^{N-1} f[k]\, e^{-2\pi i mk/N}$

$$\begin{bmatrix} f[0] \\ f[1] \\ \vdots \\ f[N-1] \end{bmatrix}$$

see the complex exponentials as discrete signals

$\omega = \left(1, e^{2\pi i \frac{1}{N}}, \cdots, e^{2\pi i \frac{N-1}{N}}\right) \qquad \omega[m] = e^{2\pi i \frac{m}{N}}$

$\mathcal{F}f = \sum_{k=0}^{N-1} f[k]\, \omega^{-k}$

$$\begin{array}{cccccc} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ e^{-2\pi i \frac{1}{N}} \\ e^{-2\pi i \frac{2}{N}} \\ \vdots \\ e^{-2\pi i \frac{N-1}{N}} \end{bmatrix} & \begin{bmatrix} 1 \\ e^{-2\pi i \frac{2\cdot 1}{N}} \\ e^{-2\pi i \frac{2\cdot 2}{N}} \\ \vdots \\ e^{2\pi i \frac{2(N-1)}{N}} \end{bmatrix} & \cdots & \begin{bmatrix} 1 \\ e^{-2\pi i \frac{(N-1)1}{N}} \\ e^{-2\pi i \frac{(N-1)2}{N}} \\ \vdots \\ e^{2\pi i \frac{(N-1)(N-1)}{N}} \end{bmatrix} & \begin{bmatrix} \mathcal{F}f[0] \\ \mathcal{F}f[1] \\ \vdots \\ \mathcal{F}f[N-1] \end{bmatrix} \\ \omega^{-0} & \omega^{-1} & \omega^{-2} & & \omega^{-(N-1)} & \end{array} \; m$$

- orthogonality of complex exponential

$\omega = \left(1, e^{2\pi i \frac{1}{N}}, \cdots, e^{2\pi i \frac{N-1}{N}}\right)$

$\omega^k = \left(1, e^{2\pi i \frac{k}{N}}, \cdots, e^{2\pi i \frac{k(N-1)}{N}}\right)$

$\omega^l \perp \omega^k$ if $k \neq l$

$$\begin{aligned} \langle \omega^l, \omega^k \rangle &= \sum_{n=0}^{N-1} \omega^k[n] \cdot \overline{\omega^l[n]} \\ &= \sum_{n=0}^{N-1} e^{2\pi i \frac{nk}{N}} \cdot \overline{e^{2\pi i \frac{nl}{N}}} \\ &= \sum_{n=0}^{N-1} e^{2\pi i \frac{nk}{N}} \cdot e^{-2\pi i \frac{nl}{N}} \\ &= \sum_{n=0}^{N-1} e^{2\pi i (k-l) \frac{n}{N}} \end{aligned}$$

if $k \neq l \% N$ $\quad (k-l)/N$ is not an integer

$$\begin{aligned} \sum_{n=0}^{N-1} e^{2\pi i (k-l)\frac{n}{N}} &= \frac{1 - \left(e^{2\pi i (k-l)/N}\right)^N}{1 - e^{2\pi i (k-l)/N}} \\ &= \frac{1 - e^{2\pi i (k-l)}}{1 - e^{2\pi i (k-l)/N}} \Big\} \cos 2\pi(k-l) + i \sin(2\pi(k-l)) = 1 \\ &= 0 \end{aligned}$$

if $k = (l \% N)$

$$\sum_{n=0}^{N-1} e^{2\pi i (k-l)\frac{n}{N}} = n$$

$$\langle \omega^l, \omega^k \rangle = \begin{cases} 0 & l \neq (k \% N) \\ N & l = (k \% N) \end{cases}$$

the norm of $\omega^k = \sqrt{N}$ $\qquad$ orthogonal, but not orthonormal

• Inverse DFT

$$\mathcal{F}^{-1} f[m] = \frac{1}{N}\sum_{k=0}^{N-1} f[k]\, e^{2\pi i\, mk/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} f[k]\, \omega^k[m]$$

$$\mathcal{F}^{-1} f = \frac{1}{N}\sum_{k=0}^{N-1} f[k]\, \omega^k$$

$$f = \begin{bmatrix} f[0] \\ f[1] \\ \vdots \\ f[N-1] \end{bmatrix}$$

$$W \quad \begin{array}{ccccc} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ e^{-2\pi i \frac{1}{N}} \\ e^{-2\pi i \frac{2}{N}} \\ \vdots \\ e^{-2\pi i \frac{N-1}{N}} \end{bmatrix} & \begin{bmatrix} 1 \\ e^{-2\pi i \frac{2\cdot 1}{N}} \\ e^{-2\pi i \frac{2\cdot 2}{N}} \\ \vdots \\ e^{-2\pi i \frac{2(N-1)}{N}} \end{bmatrix} & \cdots & \begin{bmatrix} 1 \\ e^{-2\pi i \frac{(N-1)1}{N}} \\ e^{-2\pi i \frac{(N-1)2}{N}} \\ \vdots \\ e^{-2\pi i \frac{(N-1)(N-1)}{N}} \end{bmatrix} \\ \omega^{-0} & \omega^{-1} & \omega^{-2} & & \omega^{-(N-1)} \end{array} \qquad \begin{bmatrix} \mathcal{F}f[0] \\ \mathcal{F}f[1] \\ \vdots \\ \mathcal{F}f[N-1] \end{bmatrix} \ \mathcal{F}f$$

$$W^T = N W^{-1} = N\overline{W} \qquad \begin{array}{c} \omega^0 \\ \omega^1 \\ \omega^2 \\ \\ \omega^{N-1} \end{array} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{2\pi i \frac{1}{N}} & e^{2\pi i \frac{2}{N}} & \cdots & e^{2\pi i \frac{N-1}{N}} \\ 1 & e^{2\pi i \frac{2\cdot 1}{N}} & e^{2\pi i \frac{2\cdot 2}{N}} & \cdots & e^{2\pi i \frac{2(N-1)}{N}} \\ & & \vdots & & \\ 1 & e^{2\pi i \frac{(N-1)\cdot 1}{N}} & e^{2\pi i \frac{(N-1)\cdot 2}{N}} & \cdots & e^{2\pi i \frac{(N-1)(N-1)}{N}} \end{bmatrix} \qquad \begin{bmatrix} N\cdot f[0] \\ N\cdot f[1] \\ \\ \\ N\cdot f[1] \end{bmatrix} \ Nf$$

$$W = \begin{bmatrix} \text{---}\ \omega^0\ \text{---} \\ \text{---}\ \omega^1\ \text{---} \\ \vdots \\ \text{---}\ \omega^{N-1}\ \text{---} \end{bmatrix}$$ has orthogonal rows and columns

$$W\overline{W} = NI$$

It's all about this matrix multiplication, nothing more