## · Derivative of distributions

$T$ is a given distribution, to define the derivative $T'$, have to define the pairing $\langle T', \phi\rangle$

what will happen if $T'$ is a function and $\langle T', \phi\rangle$ is given by integration

$$\begin{aligned}\langle T', \phi\rangle &= \int_{-\infty}^{+\infty} T'(x)\,\phi(x)\,dx \\ &= T(x)\,\phi(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} T(x)\,\phi'(x)\,dx \\ &= -\int_{-\infty}^{+\infty} T(x)\,\phi'(x)\,dx \\ &= -\langle T, \phi'\rangle\end{aligned}$$

( $\phi$ is a test function $\phi(x)=0$ as $x\to\pm\infty$, $T$ is rapidly decreasing function )

this makes sense even if the intermediate steps didn't

turn $\langle T', \phi\rangle$ into a definition

define $T'$ by $\langle T', \phi\rangle = -\langle T, \phi'\rangle$

eg: step function

$$u(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0\end{cases}$$

$u$ defines a distribution

$$\begin{aligned}\langle u, \phi\rangle &= \int_{-\infty}^{+\infty} u(x)\,\phi(x)\,dx \\ &= \int_{0}^{+\infty} \phi(x)\,dx\end{aligned}$$

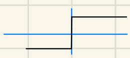
makes sense if $\phi(x)$ is a rapidly decreasing function

$$\begin{aligned}\langle u', \phi\rangle &= -\langle u, \phi'\rangle = -\int_{-\infty}^{+\infty} u(x)\,\phi'(x)\,dx \\ &= -\int_{0}^{+\infty} \phi'(x)\,dx = -\phi(x)\Big|_0^{+\infty} \\ &= \phi(0) = \langle \delta, \phi\rangle\end{aligned}$$

$u' = \delta$

eg signum

$$u(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0\end{cases}$$



$$\begin{aligned}\langle u', \phi\rangle &= -\langle u, \phi'\rangle \\ &= -\int_{-\infty}^{+\infty} u(x)\,\phi'(x)\,dx = -\left[\int_{-\infty}^{0} -\phi'(x)\,dx + \int_{0}^{+\infty} \phi'(x)\,dx\right] \\ &= \phi(x)\Big|_{-\infty}^{0} - \phi(x)\Big|_0^{+\infty} = \phi(0) - \phi(-\infty) - \phi(+\infty) + \phi(0) \\ &= 2\,\phi(0) = \langle 2\delta, \phi\rangle\end{aligned}$$

## · Application to Fourier Transform

$$\begin{aligned}\langle (\mathcal{F}T)', \phi\rangle &= -\langle \mathcal{F}T, \phi'\rangle = -\langle T, \mathcal{F}(\phi')\rangle \\ &= -\int_{-\infty}^{+\infty} T(x)\,\mathcal{F}(\phi')(x)\,dx = -\int_{x=-\infty}^{+\infty} T(x) \int_{u=-\infty}^{+\infty} \phi'(u)\,e^{-2\pi i x u}\,du\,dx \\ &= -\int_{-\infty}^{+\infty} T(x)\left[\phi(u)\,e^{-2\pi i x u}\Big|_{u=-\infty}^{+\infty} - (-2\pi i x)\int_{-\infty}^{+\infty} \phi(u)\,e^{-2\pi i x u}\,du\right]dx \\ &= -\int_{-\infty}^{+\infty} T(x)\cdot 2\pi i x\,(\mathcal{F}\phi)(x)\,dx \\ &= \langle -T(x)\cdot 2\pi i x, \mathcal{F}\phi\rangle \\ &= \langle \mathcal{F}(-T(x)\,2\pi i x), \phi\rangle\end{aligned}$$

$$(\mathcal{F}T)' = \mathcal{F}(-T(x)\,2\pi i x)$$

$$\begin{aligned}\langle \mathcal{F}(T'), \phi\rangle &= \langle T', \mathcal{F}\phi\rangle = -\langle T, (\mathcal{F}\phi)'\rangle \\ &= -\int_{-\infty}^{+\infty} T(x)\,\frac{d}{dx}\int_{-\infty}^{+\infty} \phi(u)\,e^{-2\pi i x u}\,du\,dx \\ &= -\int_{-\infty}^{+\infty} T(x)\int_{-\infty}^{+\infty} -2\pi i u\,\phi(u)\,e^{-2\pi i x u}\,du\,dx \\ &= \int_{-\infty}^{+\infty} T(x)\,\mathcal{F}(2\pi i u\,\phi(u))(s)\,dx \\ &= \langle T, \mathcal{F}(2\pi i u\,\phi(u))\rangle \\ &= \langle \mathcal{F}T, 2\pi i u\,\phi(u)\rangle \\ &= \langle 2\pi i s\cdot(\mathcal{F}T)(s), \phi\rangle\end{aligned}$$

same form as classical case

$$\mathcal{F}(T') = 2\pi i s\,(\mathcal{F}T)(s)$$

eg. F.T of the signum function

$\text{signum}' = 2\delta$

$$\left.\begin{aligned} \mathcal{F}(\text{signum}')(s) &= 2\pi i s\, \mathcal{F}(\text{signum})(s) \\ \mathcal{F}(\text{signum}')(s) &= 2\cdot \mathcal{F}\delta(s) = 2 \end{aligned}\right\} \quad \mathcal{F}(\text{signum})(s) = \frac{1}{\pi i s}$$

$$\langle \mathcal{F}\,\text{signum}, \phi\rangle = \langle \text{signum}, \mathcal{F}\phi\rangle = \int_{-\infty}^{+\infty} \text{signum}\cdot \mathcal{F}\phi(x)\,dx$$

$$= -\int_{-\infty}^{0} \mathcal{F}\phi(x)\,dx + \int_{0}^{+\infty} \mathcal{F}\phi(x)\,dx$$

$$\underset{u=-x}{=} -\int_{+\infty}^{0} \mathcal{F}\phi(-u)\,du + \int_{0}^{+\infty} \mathcal{F}\phi(x)\,dx$$

$$= \int_{0}^{+\infty} -\mathcal{F}\phi(-x)\,dx + \int_{0}^{+\infty} \mathcal{F}\phi(x)\,dx$$

$$= \int_{0}^{+\infty} \mathcal{F}\phi(x) - \mathcal{F}\phi(-x)\,dx$$

$$= \int_{0}^{+\infty}\!\!\int_{0}^{+\infty} \phi(u)\left[e^{-2\pi i x u} - e^{2\pi i x u}\right] du\,dx$$

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eg. F.T of unit step

$\text{step}(x) = \frac{1}{2}(\text{signum}(x) + 1)$

$$\mathcal{F}\text{step} = \frac{1}{2}\mathcal{F}(\text{signum} + 1)$$

$$= \frac{1}{2}\left[\mathcal{F}(\text{signum}) + \mathcal{F}(1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\pi i s} + \mathcal{F}\mathcal{F}\delta\right] = \frac{1}{2}\left[\frac{1}{\pi i s} + \delta^{-}\right]$$

$$= \frac{1}{2}\left(\frac{1}{\pi i s} + \delta\right)$$

## · Multiplication

multiplication of functions does not carry over to multiplication of distributions

( $T_1$ and $T_2$ are distributions, $T_1 \cdot T_2$ is generally not defined )

$fT$ is defined where $f$ is a function

if $T$ is given by a function, then define

$$\langle fT, \phi\rangle = \int_{-\infty}^{+\infty} f(x)T(x)\phi(x)\,dx = \langle T, f\phi\rangle \qquad \text{when } f\phi \text{ is a test function}$$

special case $f\cdot\delta$

$$\langle f\delta, \phi\rangle = \langle \delta, f\phi\rangle = f(0)\phi(0) = \langle f(0)\delta, \phi\rangle$$

$$f\delta = f(0)\delta \qquad ( f\delta_a = f(a)\delta_a )$$

## · Convolution

if $T_1$ and $T_2$ are distributions, the convolution $T_1 * T_2$ is not always defined

need some restrictions on $T_1$ and $T_2$

can define $T_1 * T_2$ by pairing, but need extra conditions

many cases it works fine and the convolution theorem holds $\mathcal{F}(f * T) = (\mathcal{F}f)\cdot(\mathcal{F}T)$

eg. $f*T$ convolving a function with a distribution

$$\langle f*T, \phi\rangle = \int_{-\infty}^{+\infty} (f*T)(x)\,\phi(x)\,dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x-y)T(y)\,dy\,\phi(x)\,dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x-y)\,\phi(x)\,dx\,T(y)\,dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f^{-}(y-x)\,\phi(x)\,dx\,T(y)\,dy$$

$$= \langle T, f^{-} * \phi\rangle$$

$$\langle f*T, \phi\rangle = \langle T, f^{-} * \phi\rangle \qquad \text{if } f \text{ is a function and } T \text{ is a distribut}$$

eg. convolve with $\delta$

$$\langle f*\delta, \phi\rangle = \langle \delta, f^- * \phi\rangle = (f^- * \phi)(0)$$

$$= \int_{-\infty}^{+\infty} f^-(-y+0)\,\phi(y)\,dy = \int_{-\infty}^{+\infty} f(y)\,\phi(y)\,dy$$

$$= \langle f, \phi\rangle$$

$f*\delta = f$ $\quad (f*\delta_a)(x) = f(x-a)$

Scaling property of $\delta$ $\quad \delta(ax)$

$$\langle \delta(ax), \phi(x)\rangle = \int_{-\infty}^{+\infty} \delta(ax)\,\phi(x)\,dx$$

$$= \int_{-\infty}^{+\infty} \delta(u)\,\phi(u/a)\,d\,u/a$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} \delta(u)\,\phi(u/a)\,du$$

$$= 1/a\,\langle \delta, \phi(u/a)\rangle$$

$$= 1/a\,\phi(0)$$

$$= 1/a\,\langle \delta, \phi\rangle \quad (a>0)$$

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$