· **Fourier Transform of a distribution / generalized function**

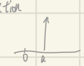
1). define a class of test functions, which typically have particularly nice property

(rapidly decreasing function for F.T)

2). a distribution T is a continuous linear functional on test functions (i.e. $T(\phi_1)+T(\phi_2) = T(\phi_1+\phi_2)$ $T(\phi_n) \to T(\phi)$ if $\phi_n \to \phi$)
$T(c\phi) = cT(\phi)$

the class of distributions is the dual space of test function

eg. $\langle \delta, \phi \rangle = \phi(0)$ conventional picture of δ_a



eg. distribution induced by functions

if $f(x)$ is a function s.t $\int_{-\infty}^{+\infty} f(x)\,\phi(x)\,dx$ converges since $\phi(x)$ has good property (rapidly decreasing), f can include many functions like $\sin$, $\cos$, 1

then define $\langle f, \phi \rangle = \int_{-\infty}^{+\infty} f(x)\phi(x)\,dx$ here f is not a function, it's a distribution induced by function f

3). Fourier Transform of distributions

let test functions to be $\mathcal{A}$ rapidly decreasing functions (remember that if $f \in$. then $\mathcal{F}f \in \mathcal{A}$, inverse F.T works $\mathcal{F}^{-1}f \in \mathcal{A}$)

the corresponding class of distributions is called the class of tempered distribution

if T is a tempered distribution, want to define the F.T $\mathcal{F}T$ to be another tempered distribution

have to define $\langle \mathcal{F}T, \phi \rangle$

suppose $\langle \mathcal{F}T, \phi \rangle$ is defined by integration (if everything is nice)

$$\begin{aligned}
\langle \mathcal{F}T, \phi \rangle &= \int_{-\infty}^{+\infty} \mathcal{F}T(x)\,\phi(x)\,dx \\
&= \int_{x} \int_{-\infty}^{+\infty} \int_{y} \int_{-\infty}^{+\infty} e^{-2\pi i x y}\, T(y)\,dy\,\phi(x)\,dx \\
&= \int_{y} \int_{-\infty}^{+\infty} \int_{x} \int_{-\infty}^{+\infty} e^{-2\pi i x y}\,\phi(x)\,dx\,T(y)\,dy \\
&= \int_{y} \int_{-\infty}^{+\infty} \mathcal{F}\phi(y)\,T(y)\,dy \\
&= \langle T, \mathcal{F}\phi \rangle
\end{aligned}$$

$$\langle \mathcal{F}^{-1}T, \phi \rangle = \langle T, \mathcal{F}^{-1}\phi \rangle$$

T is a tempered distribution that operates on rapidly decreasing functions

if ϕ is a rapidly decreasing function, then so is $\mathcal{F}\phi$

thus $\langle T, \mathcal{F}\phi \rangle$ makes sense

turn solution into a definition

for a tempered distribution T, define $\mathcal{F}T$ by $\langle \mathcal{F}T, \phi \rangle = \langle T, \underbrace{\mathcal{F}\phi}_{\text{classical F.T, well defined for rapidly decreasing } \phi} \rangle$

$$\langle \mathcal{F}^{-1}\mathcal{F}T, \phi \rangle = \langle \mathcal{F}T, \mathcal{F}^{-1}\phi \rangle = \langle T, \mathcal{F}\mathcal{F}^{-1}\phi \rangle = \langle T, \phi \rangle$$

thus for a tempered distribution T, $\mathcal{F}^{-1}\mathcal{F}T = T$

eg. Calculate the Fourier Transform of δ

$$\langle \mathcal{F}\delta, \phi \rangle = \langle \delta, \mathcal{F}\phi \rangle$$
$$= \mathcal{F}\phi(0) = \int_{-\infty}^{+\infty} e^{-2\pi i 0 x} \phi(x)\, dx$$
$$= \int_{-\infty}^{+\infty} 1 \cdot \phi(x)\, dx = \langle 1, \phi \rangle$$

$\mathcal{F}\delta = 1$ dual relationship : concentrated in time domain spreaded out in frequency domain

eg. δ_a

$$\langle \mathcal{F}\delta_a, \phi \rangle = \langle \delta_a, \mathcal{F}\phi \rangle$$
$$= \mathcal{F}\phi(a) = \int_{-\infty}^{+\infty} e^{-2\pi i a x} \phi(x)\, dx$$
$$= \langle e^{-2\pi i a x}, \phi \rangle$$

$\mathcal{F}\delta_a = e^{-2\pi i a x}$

eg. $\mathcal{F}(e^{2\pi i a x})$

$$\langle \mathcal{F} e^{2\pi i a x}, \phi \rangle = \langle e^{2\pi i a x}, \mathcal{F}\phi \rangle$$
$$= \int_{-\infty}^{+\infty} e^{2\pi i a x} \cdot \mathcal{F}\phi(x)\, dx$$
$$= [\mathcal{F}^{-1}(\mathcal{F}\phi)](a) = \phi(a)$$
$$= \langle \delta_a, \phi \rangle$$

$\mathcal{F} e^{2\pi i a x} = \delta_a$

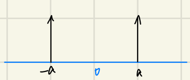
when $a=0$ $\langle \mathcal{F}1, \phi \rangle = \langle 1, \mathcal{F}\phi \rangle$

$$= \int_{-\infty}^{+\infty} \mathcal{F}\phi(x)\, dx = \int_{-\infty}^{+\infty} e^{2\pi i 0 x} \mathcal{F}\phi(x)\, dx$$
$$= (\mathcal{F}^{-1}\mathcal{F}\phi)(0) = \phi(0) = \langle \delta, \phi \rangle$$

$\mathcal{F} e^{2\pi i 0 x} = \delta_0$

eg. $\cos 2\pi a x$

$$\cos 2\pi a x = \frac{1}{2}\left(e^{2\pi i a x} + e^{-2\pi i a x}\right)$$
$$\mathcal{F}\cos 2\pi a x = \frac{1}{2}\left(\mathcal{F}e^{2\pi i a x} + \mathcal{F}e^{-2\pi i a x}\right)$$
$$= \frac{1}{2}\left(\delta_a + \delta_a\right)$$



$$\mathcal{F}\sin 2\pi a x = \frac{1}{2i}\left(\delta_a - \delta_{-a}\right)$$