· Solving those problems

Identify the "best functions" for Fourier Transform

call the class of signals $\mathcal{S}$ (pronounce "S", stands for schwarz)

want
1. if $f \in \mathcal{S}$
   then $\mathcal{F}f$ is well defined by the integral and $\mathcal{F}f \in \mathcal{S}$ (rules out many functions)
2. Inverse Fourier Transform works ($\mathcal{F}^{-1}\mathcal{F}f = f$ $\quad \mathcal{F}\mathcal{F}^{-1}f = f$)
3. Further property: parseval's identity
$$\int_{-\infty}^{+\infty} |\mathcal{F}f(s)|^2\, ds = \int_{-\infty}^{+\infty} |f(t)|^2\, dt \qquad \text{(Rayleigh's Identity for F.T)}$$

Define $\mathcal{S}$ (solved by Laurent Schwartz)

$\mathcal{S}$ is class of rapidly decreasing functions

1° any $f \in \mathcal{S}$ is infinitely differentiable

2° for any integers $m, n \geq 0$ $\quad |x|^n \cdot \left|\frac{d^m f}{dx^m}\right| \rightsquigarrow 0$ as $x \rightsquigarrow \pm\infty$

any derivative of f tends to 0 faster than any power of x

$\mathcal{S}$ has excluded many functions, to make it more general, pick up another line of development.

idea of "generalized functions", also known as "distributions" (not related to probability distribution)

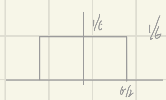
· Delta function $\delta(x)$

define $\delta(x) = \begin{cases} \text{(a).} & \begin{cases} \delta(x) = 0 & x \neq 0 \\ \delta(x) = +\infty & x = 0 \end{cases} \\ \text{(b)} & \int_{-\infty}^{+\infty} \delta(x)\, dx = 1 \\ \text{(c)} & \int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = f(0) \end{cases}$

$\delta(x)$ is supposed to represent a function which is concentrated at a point

(varying ways to do this, but always via limit)

eg. $\frac{1}{\epsilon}\Pi_\epsilon(x)$ $\quad$ as $\epsilon \to 0$ $\quad \Pi_\epsilon(x) \to \delta(x)$ $\qquad \frac{1}{\epsilon} \cdot \Pi_\epsilon(x)$



$$\int_{-\infty}^{+\infty} \frac{1}{\epsilon}\Pi_\epsilon(x) = \int_{-\epsilon/2}^{\epsilon/2} \frac{1}{\epsilon}\, dx = 1$$

$$\frac{1}{\epsilon}\int_{-\infty}^{+\infty} \Pi_\epsilon(x) \cdot f(x)\, dx = \frac{1}{\epsilon}\int_{-\epsilon/2}^{\epsilon/2} f(x)\, dx$$

$$= \frac{1}{\epsilon}\int_{-\epsilon/2}^{\epsilon/2} \underbrace{f(0) + f'(0)\cdot x + \frac{1}{2}f''(x)x^2 \cdots}_{\text{Taylor}}\, dx$$

$$= \frac{1}{\epsilon}\left[\epsilon f(0) + f'(0)\int_{-\epsilon/2}^{\epsilon/2} x\, dx + \cdots\right]$$

$$= f(0) + \frac{1}{\epsilon}\left[f'(0)\int_{-\epsilon/2}^{\epsilon/2} x\, dx + \cdots\right]$$

$$= f(0) + \frac{1}{\epsilon}\left[O(\epsilon^2) \cdots\right]$$

$$\lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} \frac{1}{\epsilon}\Pi_\epsilon(x) f(x)\, dx = f(0)$$

$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\Pi_\epsilon(x)$ makes no sense

$\lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} \frac{1}{\epsilon}\Pi_\epsilon(x) f(x)\, dx = f(0)$ does make sense change point of view: Focus on $\underbrace{\text{outcome}}_{\text{get } f(0)}$ rather than on $\underbrace{\text{the process}}_{\text{limit } \epsilon \to 0}$

·Generalized function / distributions

1) Start with "test functions" (best functions for the concerning properties. for F.T, it's the class of rapidly decreasing functions)

2) Associated with this class of "test functions" is a class of generalized functions / distributions.

   A distribution T is a linear functional on the test functions

   ↓

   for a test function ϕ, $T(\phi)$ is a (complex) number.

$$T(\phi_1+\phi_2) = T(\phi_1) + T(\phi_2)$$
$$T(c\phi_1) = c \cdot T(\phi_1)$$

3). Continuity: if a series of distribution $\phi_n \to \phi$ (convergence of a sequence of functions)

then $T(\phi_n) \to T(\phi)$ (convergence of number)

(a distribution is "paired" with a test function, written as $\langle T, \phi \rangle$

recover $\delta(x)$:

operationally, the effect of $\delta(x)$ is to pull out the value at the origin

define δ by $\langle \delta, \phi \rangle = \phi(0)$ if $\phi_n \to \phi$, then $\langle \delta, \phi_n \rangle = \phi_n(0) = \phi(0)$

define δ_a by $\langle \delta_a, \phi \rangle = \phi(a)$

$\langle \delta, \phi \rangle$

↙ distribution ↘ test function

Λ, Π, Sinusoidals come back in scene: consider "ordinary functions" in the context

eg. constant function 1 as a distribution

given a test function ϕ $\langle 1, \phi \rangle = \int_{-\infty}^{+\infty} 1 \cdot \phi(x)\, dx$

likewise, for (eg. Π), $\langle \Pi, \phi \rangle = \int_{-\infty}^{+\infty} \Pi(x)\, \phi(x)\, dx$

$$\langle \sin 2\pi x, \phi \rangle = \int_{-\infty}^{+\infty} \sin 2\pi x \; \phi(x)\, dx$$

for many function. can consider $f(x)$ as a generalized function/distribution by defining $\langle f, \phi \rangle = \int_{-\infty}^{+\infty} f(x)\, \phi(x)\, dx$

for F.T. the Schwartz functions as test functions are just the right class so that we can include sin, cos, Λ, Π ··· into distributions