Need a more robust definition of F.T to deal with common signals,
ones for which the classical definition does not work

2 issues: $\begin{cases} \text{convergence of the integral defining } \mathcal{F} \\ \mathcal{F}^{-1} \end{cases}$

· problems with F.T

problem 1: evaluating Inverse Fourier Transform

$f(t) = \Pi(t) \quad \mathcal{F}f(s) = \int_{-\infty}^{+\infty} \Pi(t)\, e^{-2\pi i s t}\, dt = \text{sinc}(s)$

$\mathcal{F}^{-1}\text{sinc} = \Pi$

last time we used duality $\quad \mathcal{F}^{-1}f = \mathcal{F}f^{-} = \mathcal{F}\,\text{sinc}^{-} = \mathcal{F}\,\text{sinc} = \mathcal{F}\mathcal{F}\Pi = \Pi^{-} = \Pi$

$(\mathcal{F}^{-1}\text{sinc})(t) = \int_{-\infty}^{+\infty} \frac{\sin \pi s}{\pi s} \cdot e^{2\pi i s t}\, ds$ can evaluate this integral, but need special technique

problem 2: evaluating Fourier Transform

$f(t) = 1$

$\mathcal{F}f(s) = \int_{-\infty}^{+\infty} e^{-2\pi i s t}\, dt$ cannot evaluate this integral

Similar: $f(t) = \sin t \quad \cos t$

· Solving those problems

Identify the "best functions" for Fourier Transform
call the class of signals $\mathcal{S}$ (pronounce "S", stands for Schwarz)

want
1. if $f \in \mathcal{S}$
   then $\mathcal{F}f$ is well defined by the integral and $\mathcal{F}f \in \mathcal{S}$ (rules out many functions)
2. Inverse Fourier Transform works ($\mathcal{F}^{-1}\mathcal{F}f = f \quad \mathcal{F}\mathcal{F}^{-1}f = f$)
3. Further property: parseval's identity
   $$\int_{-\infty}^{+\infty} |\mathcal{F}f(s)|^2\, ds = \int_{-\infty}^{+\infty} |f(t)|^2\, dt$$ (Rayleigh's Identity for F.T)

Define $\mathcal{S}$ (solved by Laurent Schwartz)

$\mathcal{S}$ is class of rapidly decreasing functions

1° any $f \in \mathcal{S}$ is infinitely differentiable

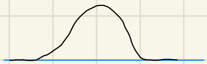
2° for any integers $m, n \geq 0 \quad |x|^n \cdot \left|\frac{d^m f}{dx^m}\right| \longrightarrow 0$ as $x \to \pm\infty$

any derivative of f tends to 0 faster than any power of x

some $f \in \mathcal{S}$:

Gaussian: $f(x) = e^{-\pi x^2}$

$\mathcal{C}$ = {infinitely differentiable functions which $=0$ outside some finite interval (different interval for different function)}



$\mathcal{C}$ stands for "compact support"

The properties of $\mathcal{S}$ come from the derivative theorem

$$\underbrace{\mathcal{F}(f^{(m)})}_{\text{differentiability}} = \underbrace{(2\pi i s)^m}_{\text{rate of decay}} \cdot \mathcal{F}f \qquad \text{(lec 9)}$$

Parseval's Identity:

for $f \in \mathcal{S}$: $\int_{-\infty}^{+\infty} |\mathcal{F}f(s)|^2 \, ds = \int_{-\infty}^{+\infty} |f(t)|^2 \, dt$

prove a more general form:

$$\int_{-\infty}^{+\infty} \mathcal{F}f(s) \cdot \overline{\mathcal{F}g(s)} \, ds = \int_{-\infty}^{+\infty} f(x) \, \overline{g(x)} \, dx$$

$$\int_{-\infty}^{+\infty} f(x) \, \overline{g(x)} \, dx$$

$$= \int_{-\infty}^{+\infty} f(x) \cdot \overline{\int_{-\infty}^{+\infty} e^{2\pi i s x} \mathcal{F}g(s) \, ds} \, dx$$

$$= \int_{-\infty}^{+\infty} f(x) \int_{-\infty}^{+\infty} e^{-2\pi i s x} \, \overline{\mathcal{F}g(s)} \, ds \, dx$$

$$= \int_{x=-\infty}^{+\infty} \int_{s=-\infty}^{+\infty} f(x) \, \overline{\mathcal{F}g(s)} \, e^{-2\pi i s x} \, ds \, dx$$

$$= \int_{s=-\infty}^{+\infty} \overline{\mathcal{F}g(s)} \int_{x=-\infty}^{+\infty} f(x) \, e^{-2\pi i s x} \, dx \, ds$$

$$= \int_{s=-\infty}^{+\infty} \overline{\mathcal{F}g(s)} \, \mathcal{F}f(s) \, ds$$

$$\int_{-\infty}^{+\infty} f(x) \, g(x) \, dx = \int_{-\infty}^{+\infty} \mathcal{F}f(s) \, \mathcal{F}g(s) \, ds$$