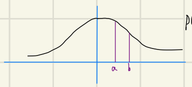
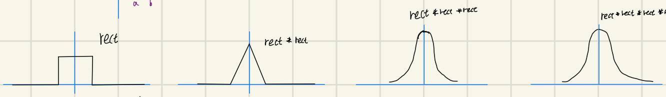
Central Limit Theorem explains the universal appearance of bell-shaped curve in probability

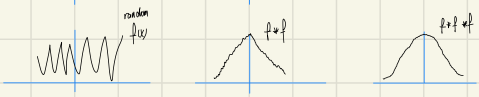
most probability, in some average sense, are approximated as if they're determined by a Gaussian




$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



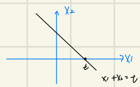

really bell-shaped




still spiky, but pretty "smooth" in big picture

the mathematical content of CLT is to explain the spikiness

$X_1 \sim p_1(x_1)$ $\quad X_2 \sim p_2(x_2)$ are 2 independent variables




$$P(x_1 + x_2 \le t) = \iint_{x_1+x_2 \le t} p_1(x_1) \cdot p_2(x_2)\, dx_1 dx_2$$

$$\overset{\substack{u = x_1 \\ v = x_1 + x_2}}{=} \int_{v=-\infty}^{t} \int_{u=-\infty}^{+\infty} p_1(u) \cdot p_2(v-u)\, du\, dv$$

$$= \int_{v=-\infty}^{t} (p_1 * p_2)(v)\, dv$$

$$p_{x_1+x_2}(x_1 + x_2) = p_1 * p_2$$

$X_1, \ldots X_n$ are iid $\quad X_i \sim p(x)$

assume $X_i$ has zero mean, unit std

$S_n = X_1 + X_2 + \cdots X_n \qquad E[S_n] = 0 \qquad \mathrm{std}(S_n) = \sqrt{n}$

$S_n / \sqrt{n}$ has mean 0, std dev

Central Limit Theorem

$$\lim_{n \to +\infty} \mathrm{Prob}\left(a \le \frac{S_n}{\sqrt{n}} \le b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx$$

An unintegrated form

$p_n(x)$ is the distribution of $\frac{x_1 + \cdots + x_n}{\sqrt{n}}$

$p_n(x) \longrightarrow \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$p^{*n}(x_1 + x_2 \cdots + x_n) = \underbrace{p * p * \cdots p}_{\text{convolves } n \text{ time}}$$

$$p_n\left(\frac{x_1 + x_2 + \cdots + x_n}{\sqrt{n}}\right) = \sqrt{n} \cdot p^{*n}(x_1 + \cdots x_n)$$

Fourier Transform Takes convolution into multiplication

$$p_n(x) = \sqrt{n} \cdot p^{*n}(\sqrt{n}\, x)$$

$$p^{*n}(x) \longrightarrow (\mathcal{F}p)^n (s)$$

$$p^{*n}(\sqrt{n}\, x) \to \frac{1}{\sqrt{n}} (\mathcal{F}p)^n (s/\sqrt{n})$$

$$\sqrt{n}\, p^{*n}(\sqrt{n}\, x) \to (\mathcal{F}p)^n (s/\sqrt{n}) = \left[\mathcal{F}p(s/\sqrt{n})\right]^n$$

$$\mathcal{F}p(s/\sqrt{n}) = \int_{-\infty}^{+\infty} e^{-2\pi i s/\sqrt{n}\, x}\, p(x)\, dx$$

$$= \int_{-\infty}^{+\infty} \Big[\underbrace{1 + (-2\pi i s/\sqrt{n})^1 x + \tfrac{1}{2}(-2\pi i s/\sqrt{n})^2 x^2 + \cdots}_{\text{taylor}}\Big]\, p(x)\, dx$$

$$= \underbrace{\int_{-\infty}^{+\infty} 1\, p(x)\, dx}_{1} - \frac{2\pi i s}{\sqrt{n}} \underbrace{\int_{-\infty}^{+\infty} x\, dx}_{0 \text{ (zero mean)}} + \frac{2\pi^2 s^2}{n} \underbrace{\int_{-\infty}^{+\infty} x^2 p(x)\, dx}_{\text{std}=1 \text{ (zero mean)}} + \cdots$$

$$= 1 - \frac{2\pi^2 s^2}{n} + \cdots$$

$$\lim_{+\infty} \left(\mathcal{F}p(s/\sqrt{n})\right)^n \approx \lim_{n\to+\infty}\left(1 - \frac{2\pi^2 s^2}{n}\right)^n$$

$$= \lim_{n\to+\infty}\left(1 + \frac{-2\pi^2 s^2}{n}\right)^{\frac{n}{-2\pi^2 s^2}\cdot -2\pi^2 s}$$

$$= e^{-2\pi^2 s^2}$$

$$\sqrt{n}\, p^{*n}(\sqrt{n}x) = \int_{-\infty}^{+\infty} e^{-2\pi^2 s^2} e^{2\pi i s x}\, ds$$

$$\underset{u=-s}{=} \int_{+\infty}^{-\infty} e^{-2\pi^2 u^2} e^{-2\pi i u x}\, d{-u}$$

$$= \int_{-\infty}^{+\infty} e^{-2\pi^2 u^2} \cdot e^{-2\pi i u x}\, du$$

$$= \left(\mathcal{F} e^{-2\pi^2 u^2}\right)(x)$$

$$e^{-\pi u^2} \longrightarrow e^{-\pi x^2}$$ Fourier Transform of Gaussian is itself

$$e^{-\pi(\sqrt{2\pi}u)^2} \longrightarrow \frac{1}{\sqrt{2\pi}} e^{-\pi\left(\frac{x}{\sqrt{2\pi}}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$ Stretched

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \mathcal{N}(x; 0, 1)$$