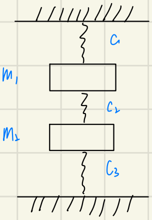
$$\begin{cases} \underbrace{M}_{\text{mass}} \cdot \underbrace{\frac{d^2u}{dt^2}}_{\text{acceleration}} + \underbrace{ku}_{\text{force on spring}} = 0 \\ u(0) = \text{starting position} \\ u'(0) = \text{starting velocity} \end{cases}$$



$$\begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \begin{bmatrix} u_1'' \\ u_2'' \end{bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$\begin{cases} \text{solve by eigenvectors} \\ \text{solve by finite differences} \end{cases}$

accuracy — eg centered difference gives 2nd order accuracy

stability

speed

$u'' + u = 0$ has eigenfunction $\sin t$ $\cos t$

$u = A\cos t + B\sin t$

· Eigenvector method

$Mu'' + ku = 0$

simplest model: 1 spring, 1 mass: $u'' + u = 0$

$u = A\cos t + B\sin t$

look for $u(t) = \underbrace{\left(\overset{\sin wt}{\cos wt}\right)}_{\text{oscillation}} \underbrace{x}_{\text{costant vector}}$ choose w and x

$e^{\lambda t}$ is "right" for 1st order equation, $\cos wt$ and $\sin wt$ are right for 2nd order equation

$-Mw^2(\cos wt)x + k \cdot (\cos wt)x = 0$

$kx = Mw^2x$ (an eigenvalue problem $M^{-1}kx = w^2x$)

$\underline{kx = w^2Mx}$ matlab $eig(k, M)$ solves $kx = w^2Mx$

generalized eigenvalue problem: expect n positive eigenvalues w^2, and n eigenvectors x

the generations $u(x) = a_1(\cos w_1t)x_1 + b_1(\sin w_1t)x_1$ a_i and b_i comes from initial states

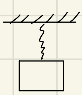
$\vdots$

$a_n(\cos w_nt)x_n + b_n(\sin w_nt)x_n$

$u(0) = a_1x_1 + \cdots a_nx_n$

$u'(0) = b_1x_1 + \cdots b_nx_n$

· Finite difference method



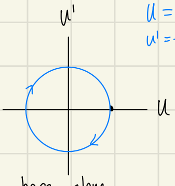
$\begin{cases} u'' + u = 0 \\ u(0) = 1 \\ u'(0) = 0 \end{cases}$

$\downarrow$

$u = \cos t$

$u' = -\sin t$

pull down and let go



phase plane

want finite difference solutions keeps on the circle

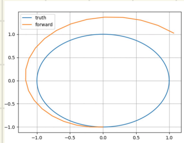
$$\begin{cases} u'' + u = 0 \\ u(0) = 0 \\ u'(0) = 1 \end{cases}$$

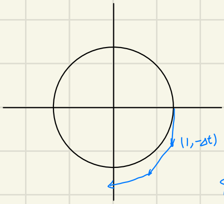
$$\frac{u_{n+1} - 2u_n + u_{n-1}}{(\Delta t)^2} = \begin{cases} - u_{n+1} & \text{backward} \\ - u_n & \text{centered} \\ - u_{n-1} & \text{forward} \end{cases}$$

$$u'' + u = 0 \rightarrow \begin{cases} u' = v \\ v' = u'' = -u \end{cases} \rightarrow \begin{cases} u' = v \\ v' = u \end{cases}$$

1° $u' = v$ $\quad u_{n+1} = u_n + \Delta t \cdot v_n$ $\quad$ forward difference

$v' = -u$ $\quad v_{n+1} = v_n - \Delta t \cdot u_n$ $\quad$ (forward Euler)





$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}$$

$\lambda_1 \lambda_2 = 1 + \Delta t^2 > 1$

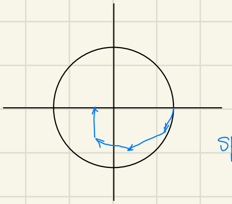
at least 1 eigenvalue bigger than 1

spirals out

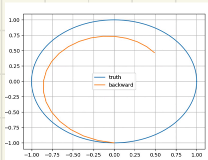
2° $u' = v$ $\quad u_{n+1} = u_n + \Delta t \, v_{n+1}$ $\quad \dfrac{u_{n+1} - u_n}{\Delta t} = u'_{n+1}$ $\quad$ backward difference

$v' = -u$ $\quad v_{n+1} = v_n - \Delta t \cdot u_{n+1}$

$$\begin{cases} u_{n+1} - \Delta t \, v_{n+1} = u_n \\ v_{n+1} + \Delta t \, u_{n+1} = v_n \end{cases} \qquad \begin{bmatrix} 1 & -\Delta t \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} u_n \\ v_n \end{bmatrix}$$

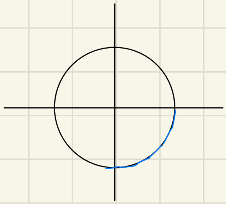


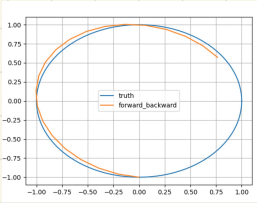
spirals in



3° $u_{n+1} = u_n + \Delta t \, v_n$ $\quad \dfrac{u_{n+1} - 2u_n + u_{n-1}}{\Delta t^2} = -u_n$

$v_{n+1} = v_n - \Delta t \, u_{n+1}$





Lec 10.

4° Trapsoidal method

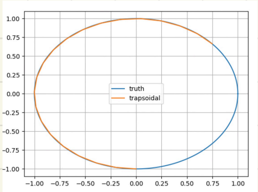
$u' = v$ $\qquad U_{n+1} = U_n + \frac{\delta t}{2}(V_n + V_{n+1})$

$v' = -u$ $\qquad V_{n+1} = V_n - \frac{\delta t}{2}(U_n + U_{n+1})$

} centered difference; second order difference

$$U_{n+1} - \frac{\delta t}{2} V_{n+1} = U_n + \frac{\delta t}{2} V_n$$
$$V_{n+1} + \frac{\delta t}{2} U_{n+1} = V_n - \frac{\delta t}{2} U_n$$

$$\begin{bmatrix} 1 & -\frac{\delta t}{2} \\ \frac{\delta t}{2} & 1 \end{bmatrix} \begin{bmatrix} U_{n+1} \\ V_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\delta t}{2} \\ -\frac{\delta t}{2} & 1 \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix}$$



$$\begin{bmatrix} U_{n+1} \\ V_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\delta t}{2} \\ \frac{\delta t}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \frac{\delta t}{2} \\ -\frac{\delta t}{2} & 1 \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix}$$

$$= \underbrace{\left(I - \frac{\delta t}{2} A\right)^{-1}\left(I + \frac{\delta t}{2} A\right)} \begin{bmatrix} U_n \\ V_n \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= G \begin{bmatrix} U_n \\ V_n \end{bmatrix}$$

$$\frac{1 + \frac{\delta t}{2}\lambda_i}{1 - \frac{\delta t}{2}\lambda_i} \qquad \lambda_i = \begin{cases} i \\ -i \end{cases}$$

$$|\mathrm{eig}(G)| = 1$$

5° BDF2 (Backward Difference Formula 2nd accurate)

$$\frac{U_{n+1} - U_n}{\delta t} + \frac{1}{2}\frac{U_{n+1} - 2U_n + U_{n-1}}{\delta t} = AU_{n+1}$$