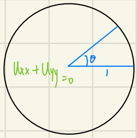
laplace equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$U(x,y) = \text{Real}\,(x+iy)^n = r^n \cos n\theta$

$S(x,y) = \text{Img}\,(x+iy)^n = r^n \sin n\theta$

all are solutions to $U_{xx} + U_{yy} = 0$



$U(1,\theta) = U_0(\theta)$ temperature on boundy (boundy condition)

$U_0(\theta)$ is periodic

Satisfies Laplace equation in the circle (no source)

eg $U_0(\theta) = \sin 3\theta$ $\qquad U(r,\theta) = r^3 \sin 3\theta$

$U(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r \sin\theta + a_2 r^2 \cos 2\theta + b_2 r^2 \sin 2\theta \ldots$

$= a_0 + \sum_n a_n r^n \cos n\theta + b_n r^n \sin n\theta$

$U(1,\theta) = a_0 + \sum_n a_n \cos n\theta + b_n \sin n\theta$ (set $r=1$ to match $U_0(\theta)$)

Fourier !!!!!

when the region is a circle, u is a smooth function, even if $U(1,\theta) = \delta(\theta)$

$U(r,\theta) = a_0 + \sum_n a_n r^n \cos n\theta + b_n r^n \sin n\theta$

when $r\downarrow$, high frequency term vanishes

$(x+iy)^n$ solves the laplace equation

$\frac{d^2}{dx^2}(x+iy)^n = n(n-1)(n-2)\,(x+iy)$

$\frac{d^2}{dy^2}(x+iy)^n = -n(n-1)(n-2)\,(x+iy)$

any $f(x+iy)$ or $f(re^{i\theta})$ solves the laplace equation

eg. $\begin{matrix}\text{Real}\\ \text{Img}\end{matrix}\; e^{x+iy} = \begin{matrix}\text{Real}\\ \text{Img}\end{matrix}\; e^x(\cos y + i \sin y) = \begin{matrix} e^x \cos y \\ e^x \sin y\end{matrix}$ solve the laplace equation

(e^z: analytic function $\qquad e^x\cos y \quad e^x \sin y$: harmonic function) eg. $\begin{matrix}\text{Real}\\ \text{Img}\end{matrix} \ln(x+iy) = \begin{matrix}\text{Real}\\ \text{Img}\end{matrix} \ln(re^{i\theta}) = \begin{matrix}\text{Real}\\ \text{Img}\end{matrix} \ln r + \ln e^{i\theta} = \begin{matrix}\text{Real}\\ \text{Img}\end{matrix} \ln r + i\theta$

$= \begin{matrix}\ln r\\ \theta\end{matrix} = \begin{matrix}\ln\sqrt{x^2+y^2} = u(x,y)\\ \arctan y/x = s(x,y)\end{matrix}$

$u(x,y)$ and $s(x,y)$ solves $\frac{d}{dx^2} + \frac{d}{dy^2} = 0$

except at $(0,0)$

$w = \text{grad}\, u = \begin{bmatrix} \partial u/dx \\ \partial u/dy \end{bmatrix}$

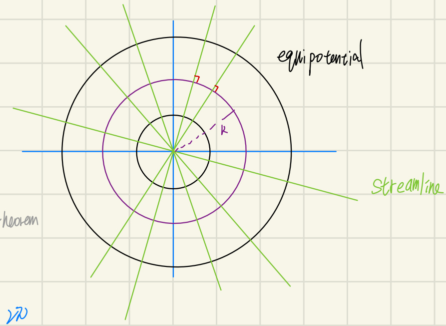
$\iint_R \text{div}\, w \, dx\,dy = \oint_R u_n \, ds$

divergence theorem

$= \oint_R d\ln R \, ds$

$= \oint_{\theta=0}^{2\pi} \frac{1}{R} \cdot R\, d\theta = 2\pi$

the source inside has "strength" 2π



$\hookrightarrow$ $u(x,y)$ and $s(x,y)$ are solutions to

possion equation $u_{xx} + u_{yy} = 2\pi \cdot \delta(0,0)$

$u = \ln r / 2\pi$: Greens function in 2D, solution to possion equation $u_{xx} + u_{yy} = \delta(0,0)$

$s = 0$

· Conformal mapping

change of coordinate to make boundary a circle (so that we can solve laplace equation by F.S)

$F(x+iy) = X(x,y) + iY(x,y)$