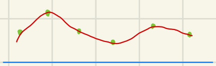
· Interpolation



1° pwl interpolation

2° $p(x) = a_0 + a_1 x_1 \cdots + a_n x_n$ unstable when n is big

3° interpolate with (cubic) splines.

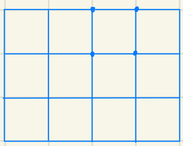
splines: no jump in function, no jump in 1st derivative, no jump in 2nd derivative

B-spline (Basic spline) could be a trial function

every spline function is some combination of B-splines B_i

$$\sum_i U_i B_i(x) = F(x)$$

for 2D function when observed points are on a grid



$$\sum_{i,j} U_i B_i(x) \, U_j B_j(x) = F(x)$$

Band-limited function (Fourier idea)

frequencies $-\Omega \le k \le \Omega$ $\quad F(x) = \int_{-\Omega}^{\Omega} c_k \cdot e^{ikx} \, dk$ } low frequency

spatial smoothness: how many derivatives are continuous

frequency smoothness: e^{ikx} for small k → low frequency.

Shannon's idea: fit a function so that it's band-limited

(Shannon band-limited stuff is the limit of splines as the spline degree goes up)

· Gradient and Divergence

temperature $u(x,y)$ $\quad$ $\text{divergence} = -\text{gradient}^T$ $\quad$ Balance equation $A^T w = -\partial w_1/\partial x - \partial w_2/\partial y = -\text{div}\, w$

$A = \text{grad}$ $\quad$ $A = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix}$ $\quad$ $A^T = [-\partial/\partial x, -\partial/\partial y]$ $\quad$ $A^T = -\text{div}$

temperature gradient $e(x,y)$ $\xrightarrow{\quad c(x,y) \quad}$ heat flow $w(x,y)$

$e(x,y) = Au = \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix}$ $\quad$ $c(x,y)$ thermal conductivity $\quad$ w is a vector

transpose of gradient

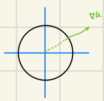
$$A = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} \qquad A^T = [\partial/\partial x^T, \partial/\partial y^T]$$
$$= [-\partial/\partial x, -\partial/\partial y]$$

· Gradient

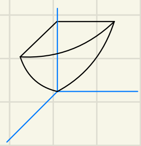
$$u(x,y) \xrightarrow{A} \text{grad}\, u = \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

① meaning of u

eg $u = x^2 + y^2$



$\text{grad}\, u = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$



steepest direction $\quad$ how steep $\|\nabla u\|_2$

perpendicular to equipotentials

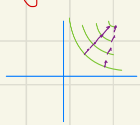
② given vector field $v_1(x,y)$ $v_2(x,y)$,
is $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ the gradient of some u?

$$\begin{cases} v_1(x,y) = \partial u/\partial x \\ v_2(x,y) = \partial u/\partial y \end{cases} \rightarrow \left\{ \begin{array}{l} \dfrac{\partial v_1}{\partial y} = \partial^2 u/\partial x \partial y \\ \dfrac{\partial v_2}{\partial x} = \partial^2 u/\partial y \partial x \end{array} \right] \text{should be same}$$

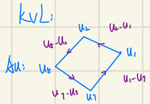
$$\text{Curl}(v) = \frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y} = 0 \quad (\text{curl in plane})$$

eg. $v = \begin{bmatrix} 2y \\ 2x \end{bmatrix}$

$u = 2xy$



KVL:



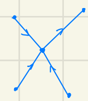
$\xrightarrow{\text{continuous}}$ $\oint v_1\, dx + v_2\, dy = 0$

· Divergence

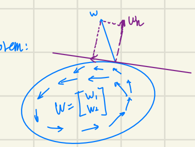
w is a vector field

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \longrightarrow \text{div}\, w = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}$$

KCL: $\nabla \cdot w = 0$



Divergence theorem:



$$\underbrace{\iint_R \left(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} \right) dx\,dy}_{\text{in \& out in the whole region}} = \underbrace{\oint_R w_n\, ds}_{\text{in \& out in the boundary}}$$

div $= 0$ at every point $\longrightarrow$ 0 across every loop