$$\begin{cases} -\dfrac{d^2u}{dx^2} = f(x) \\ u(0)=0 \\ u(1)=0 \end{cases} \Rightarrow \begin{cases} \dfrac{-U_{i+1}+2U_i-U_{i-1}}{\Delta x^2} = f(x_i) \\ U_0 = 0 \\ U_{n+1} = 0 \end{cases} \Rightarrow KU = f$$

Finite difference method

· First Difference

forward difference $\Delta_F u = \dfrac{u(x+h)-u(x)}{h} \approx u'(x) + O(h)$

backward difference $\Delta_B u = \dfrac{u(x)-u(x-h)}{h} \approx u'(x) + O(h)$

} first-order accurate

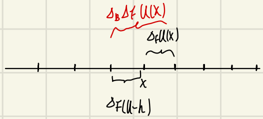
centered difference $\Delta_C u = \dfrac{u(x+h)-u(x-h)}{2h} \approx u'(x) + O(h^2)$ second-order accurate

$$\begin{cases} u(x+h) = u(x) + h\cdot u'(x) + \dfrac{h^2}{2}u''(x) + \dfrac{h^3}{6}u'''(x) + \cdots \\ u(x-h) = u(x) - h\cdot u'(x) + \dfrac{h^2}{2}u''(x) - \dfrac{h^3}{6}u'''(x) + \cdots \end{cases}$$

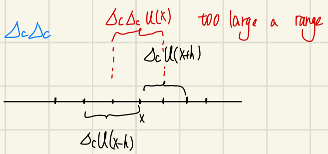
$$\begin{cases} \dfrac{u(x+h)-u(x)}{h} = u'(x) + \dfrac{h}{2}u''(x) \cdots \quad \text{first order} \quad O(h) \\ \dfrac{u(x+h)-u(x-h)}{2h} = u'(x) + \dfrac{h^2}{6}u'''(x) \cdots \quad \text{second order} \quad O(h^2) \end{cases}$$

· Second difference

$\Delta_F\Delta_B \quad \Delta_B\Delta_F$



$\Delta_C\Delta_C$



$$\Delta_F\Delta_B u(x) = \frac{\Delta_B u(x+h) - \Delta_B u(x)}{h} = \frac{\frac{u(x+h)-u(x)}{h} - \frac{u(x)-u(x-h)}{h}}{h} = \frac{u(x+h)-2u(x)+u(x-h)}{h^2}$$

$$\Delta_B\Delta_F u(x) = \frac{\Delta_F u(x) - \Delta_F(x-h)}{h} = \frac{\frac{u(x+h)-u(x)}{h} - \frac{u(x)-u(x-h)}{h}}{h} = \frac{u(x+h)-2u(x)+u(x-h)}{h^2}$$

eg. constant

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

linear

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

quadratic

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

· Differential equations

fixed
fixed

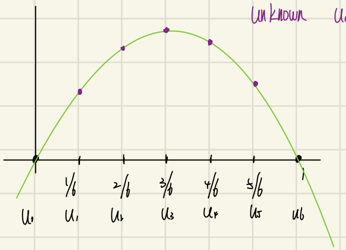
$$\begin{cases} -\frac{d^2u}{dx^2} = 1 \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

u_p to get 1 $\quad u_p(x) = -\frac{1}{2}x^2$

u_c to get 0 $\quad u_c(x) = C_1 x + C_2$

$u(x) = -\frac{1}{2}x^2 + C_1 x + C_2$, $u(0)=0$, $u(1)=0$ $\quad C_1 = 1/2$, $C_2 = 0$

unknown $u_1\ u_2\ u_3\ u_4\ u_5$



$$\begin{cases} -\frac{d^2u}{dx^2} = 1 \\ u(0) = 0 \\ u(1) = 0 \end{cases} \xrightarrow{\text{discrete}} \begin{cases} \frac{-u_{i+1} + 2u_i - u_{i-1}}{\Delta x^2} = 1 \\ u_0 = 0 \\ u_6 = 0 \end{cases}$$

$$\frac{KU}{\Delta x^2} = f \Longrightarrow \frac{1}{36} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

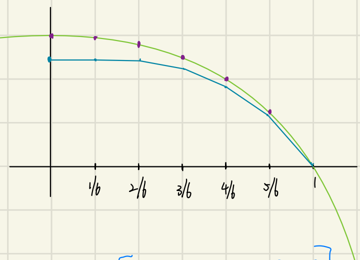
-1, $u_0 = 0$; -1, $u_6 = 0$

free - fixed

$$\begin{cases} -\frac{d^2u}{dx^2} = 1 \\ u(1) = 0 \\ u'(0) = 0 \end{cases}$$

$u(x) = -1/2\, x^2 + 1/2$

first order approximation



$$\begin{cases} -\frac{d^2u}{dx^2} = 1 \\ u(1) = 0 \\ u'(0) = 0 \end{cases} \Rightarrow \begin{cases} \frac{-u_{i+1} + 2u_i - u_{i-1}}{\Delta x^2} = 1 \\ u_6 = 0 \\ \frac{u_1 - u_0}{\Delta x} = 0 \end{cases} \quad \text{or} \quad \frac{u_1 - u_{-1}}{2\Delta x} = 0$$

first order approximation on the left boundary

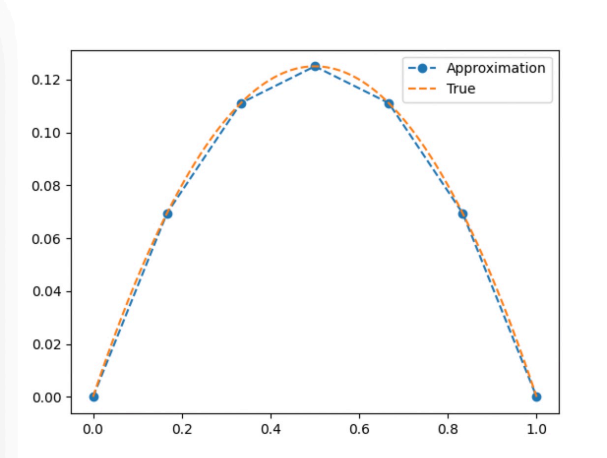
$$\frac{Tu}{\Delta x^2} = f \quad \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

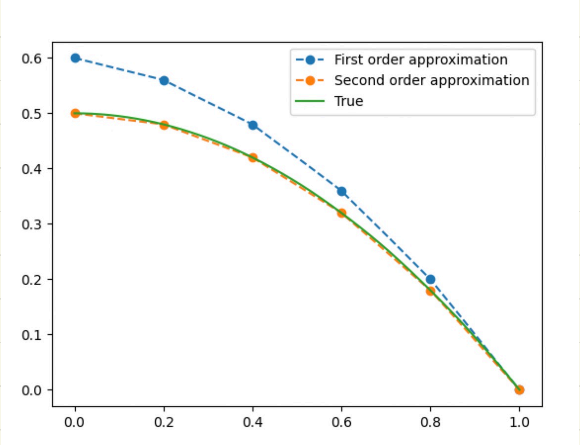
-1, $u_6 = 0$

can divide the 1st row by 2 to make the matrix symmetric

u_{-1}

$$-1 \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = 1 \xrightarrow{u_{-1} = u_1} \begin{bmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = 1$$


Approximation
True
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0.0
0.2
0.4
0.6
0.8
1.0


First order approximation
Second order approximation
True
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0