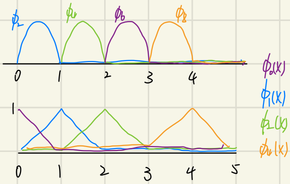
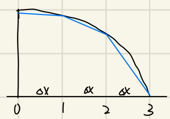
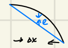
· Quadratic elements

piece-wise linear $\Rightarrow$ piece-wise quadratic







$e \sim O(\Delta x^2)$ $\Big]$ error of piece-wise linear elements

$e \sim O(\Delta x^3)$ error of quadratic elements

$$F(x) \approx \underbrace{\underbrace{F(0) + F'(0)\,x}_{\text{hat elements}} + \frac{F''(0)}{2}x^2}_{\text{quadratic elements}} + \cdots$$

weak form: $\int_0^1 c(x)\,\frac{du}{dx}\frac{dv}{dx} = \int_0^1 f(x)v(x)\,dx$

$$\downarrow$$

$$\int_0^1 c(x)\left(u_0\phi_0'(x) + \cdots \; u_8\phi_8'(x)\right)\frac{dv_i}{dx}\,dx = \int_0^1 f(x)\,v_i(x)\,dx = F_i$$

$$\downarrow$$

$$Ku = f$$

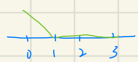
$$K_{ij} = \int_0^1 c(x)\,\frac{d\phi_j}{dx}\cdot\frac{dv_i}{dx}\,dx$$

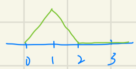
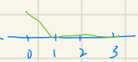
$$K_{ij} = \int_0^1 c(x)\,\frac{d\phi_j}{dx}\frac{dv_i}{dx}\,dx = \int_0^1 c(x)\,\frac{d\phi_j}{dx}\frac{d\phi_i}{dx}\,dx \quad (\text{taking } v_i = \phi_i)$$

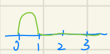
$\downarrow$ derivative of quadratic is linear

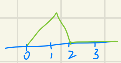
K symmetric (actually pd)

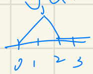
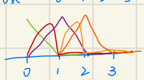
$K[1,1] = \int_0^1 c(x)\,\frac{d}{dx}$  $\frac{d}{dx}$  dx

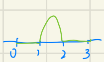
$K[1,2] = \int_0^1 c(x)\,\frac{d}{dx}$  $\frac{d}{dx}$  dx

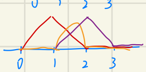
$K[1,3] = \int_0^1 c(x)\,\frac{d}{dx}$  $\frac{d}{dx}$  dx

$K[2,:] = \int_0^1 c(x)\,\frac{d\phi_j}{dx}\,\frac{d}{dx}$  dx

$K[3,:] = \int_0^1 c(x)\,\frac{d\phi_j}{dx}\,\frac{d}{dx}$  dx

(overlap with ) $\hookrightarrow$ 

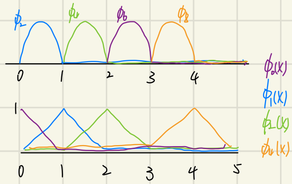
$K[4,:] = \int_0^1 c(x)\,\frac{d\phi_j}{dx}\,\frac{d}{dx}$ 

$\hookrightarrow$ 

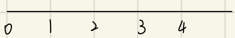
$$\begin{bmatrix} x & x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x & 0 & 0 & 0 \\ 0 & 0 & x & x & x & 0 & 0 & 0 \\ 0 & 0 & x & x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & x & x & x & 0 \\ 0 & 0 & 0 & 0 & x & x & x & 0 \end{bmatrix} \quad \int_0^1 c(x)\,\frac{d\phi_j}{dx}\,\frac{d}{dx}$$  dx

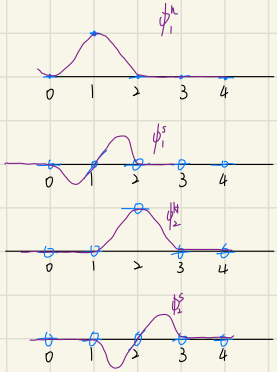
some kind of block-diagonal structure

· Cubic elements



jump in slope





gives height at dx

gives slope at dx

$a_0 + a_1 x_1 + a_2 x^2 + a_3 x^3$

9 cubic trial functions
(4 is a fixed end)
ϕ_i^h and ϕ_i^s $\quad i = 0,1,2,3$

$K \in \mathbb{R}^{9\times 9}$ six-diagonal

Strong form: $-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x)$ $\qquad A^T C A u = f$

weak form: $\int_0^1 c(x)\frac{du}{dx}\frac{dv}{dx}\,dx = \int_0^1 f(x)\,v(x)\,dx$ for all test functions $v(x)$ $\qquad \langle A^T C A u, v\rangle = \langle f, v\rangle$

$\downarrow$

$c(Au)(Av)$ $\qquad \langle CAu, Av\rangle = \langle f, v\rangle$ discrete analogy of integration by parts

minimum form: $Ku = f$

$\downarrow$

$\min_u \frac{1}{2}u^T K u - u^T f$

$\downarrow$

$\min_u \frac{1}{2}u^T A^T C A u - u^T f$

$\downarrow$

$\min_u \frac{1}{2}\int_0^1 c(x)\frac{du}{dx}\frac{du}{dx}\,dx - \int_0^1 f(x)\,u(x)\,dx$