$$-\frac{d}{dx}\left(c(x)\,\frac{du}{dx} \right) = f(x)$$

$u(x)$: displacement

$\downarrow \quad d/dx \quad (A)$

$e(x) = \frac{du}{dx}$: stretching

$\downarrow \quad c(x) \quad (C)$

$w(x) = c(x)\cdot e(x)$: Hooke's law

$\downarrow \quad -\frac{d}{dx} \quad (A^T)$

$$\left(\frac{d}{dx}\right)^T = -\frac{d}{dx}$$

with boundry conditions

Centered difference: $A = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$

$$\frac{d}{dx} = A \qquad -\frac{d}{dx} = -A = A^T$$

Define transpose by:

$\langle Au, w\rangle = \langle u \cdot A^T w\rangle \qquad (Au)^T w = u^T (A^T w)$

need inner product of 2 functions : $\langle e, w\rangle = \int_0^1 e(x)\cdot w(x)\,dx$ — integrate over the region of the problem

$$\left\langle \frac{du}{dx}, w \right\rangle = \int_0^1 \frac{du}{dx} w(x)\,dx = \int_0^1 u(x)\left(-\frac{dw}{dx}\right)dx + u(x)\cdot w(x)\Big|_0^1$$

$$\int_0^1 w(x)\,du(x) = w(x)\cdot u(x)\Big|_0^1 - \int_0^1 u(x)\cdot dw(x) = w(x)\cdot u(x)\Big|_0^1 - \int_0^1 u(x)\frac{dw}{dx}\,dx$$

eg. free-fixed

$$\frac{du}{dx} = f(x) \qquad u(1) = 0$$

$$\begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix} (1) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} u_5 = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \qquad (1)\begin{bmatrix} -1 & & & \\ 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \end{bmatrix} \begin{array}{c} w_0 \\ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \end{array}$$

$Au = f \qquad u_5 = 0$

fixed end at the right

$A^T w = \qquad w_0 = 0$

free end at the left

Continuous free-fixed

$u(1) = 0$ essential/dirichlet boundry condition — $u(1)$ decides $w(0)$

$w(0) = 0$ natural/Neumann boundry condition

$u(x)\cdot w(x)\Big|_0^1 = 0$

Square A vs. rectangular A

Square A: $(A^T C A)^{-1} = A^{-1} C^{-1} A^{-T}$ can solve backward step by step (Statically determinate)

rectangular: indeterminate

$A^T w = f(x)$ $\quad \begin{cases} -\frac{dw}{dx} = f(x) \\ w(0) = 0 \end{cases}$ can solve directly on $u(x)$

- weak form of differential equation

Strong form: $-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x)$

$$\int_0^1 -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) v(x)\,dx = \int_0^1 f(x)\cdot v(x)\,dx$$

if this is true for any $v(x)$ (with $v=0$ at fixed end) then the strong form is true $\quad$ eg. $v(x) = \begin{cases} 1 & x= \\ 0 & \text{else} \end{cases}$

$$\downarrow$$

$$\int_0^1 -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) v(x)\,dx$$

$$= -\int_{0}^{1} v(x)\, d\left(c(x)\frac{du}{dx}\right) = -v(x)\cdot c(x)\cdot\frac{du}{dx}\Big|_0^1 + \int_0^1 c(x)\cdot\frac{du}{dx}\,dv$$

$$= -v(x)\cdot c(x)\cdot\frac{du}{dx}\Big|_0^1 + \int_0^1 c(x)\cdot\frac{du}{dx}\cdot\frac{dv}{dx}\cdot dx$$

free end: $\frac{du}{dx} = 0$

fixed end $u(x) = 0$

weak form: $\int_0^1 c(x)\,\frac{du}{dx}\cdot\frac{dv}{dx}\cdot dx = \int_0^1 f(x)\cdot v(x)\,dx$ for any $v(x)$

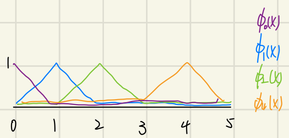
- Galerkin's method

continuous $\rightarrow$ discrete $\quad KU = F$

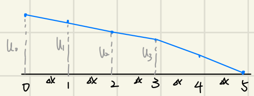
choose trial functions $\phi_1(x) \cdots \phi_n(x)$

approximate solutions $U = u_1\phi_1 + \cdots + u_N\phi_N$

get N equations by choosing N test equations $v_1 \cdots v_N$



ϕ's $= v$'s are piece-wise linear



u_i: displacement in nodes

$U(x) = u_0\,\phi_0(x) + \cdots + u_i\,\phi_i(x)$

free-fixed case $\begin{cases} \text{free} & \phi_0(0) \neq 0 \\ \text{fixed} & \phi_i(5) = 0 \ \ \forall i \end{cases}$

weak form: $\int_0^1 c(x)\,\frac{du}{dx}\,\frac{dv}{dx}\,dx = \int_0^1 f(x)\,v(x)\,dx \quad \forall v(x)$

$$\downarrow$$

$U = u_0\,\phi_0(x) + \cdots$

$\frac{du}{dx} = u_0\cdot\phi_0'(x) + \cdots$

$$\int_0^1 c(x)\left(u_0\phi_0'(x) \cdots u_4\phi_4'(x)\right)\cdot\frac{dv_i}{dx}\cdot dx = \int_0^1 f(x)\,v_i(x)\,dx = F_i$$ test against 5 v's

5 unknowns: $u_0 \cdots u_4$

test against 5 v's

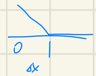
$\Big\}$ 5x5 system

eg. when $f(x) = 1$ $C(x) = 1$

$$F_0 = \int_0^1 f(x)\, V_0(x)\, dx = \int_0^1 \phi_0(x)\, dx = \tfrac{1}{2}\Delta x$$

$$F_1 = \int_0^1 f(x)\, V_1(x)\, dx = \int_0^1 \phi_1(x)\, dx = 1 \cdot \Delta x$$

$$F = [\,1/2 \;\; 1 \;\; 1 \;\; 1 \;\; 1\,] \cdot \Delta x$$

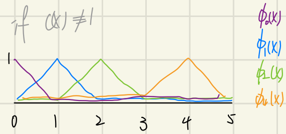


$$\int_0^1 C(x)\left(u_0 \phi_0'(x) + \cdots u_4 \phi_4'(x)\right) \frac{dv_0}{dx}\, dx = F_0$$

$$\begin{bmatrix} & & \\ & K & \\ & & \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \\ F \\ \\ \end{bmatrix}$$

approximate $\int_0^1 c(x)\, \phi_0'(x)\, v_0'\, dx$ if $c(x) \neq 1$

$$K[0,0] = \int_0^1 c(x)\, \phi_0'\, v_0'\, dx = \int_0^1 \phi_0'\, v_0'\, dx$$

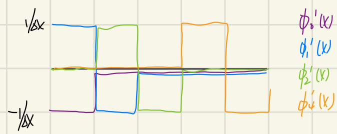


$$\phi_0'(x) = \begin{cases} -1/\Delta x & 0 \le x \le \Delta x \\ 0 & \text{else} \end{cases}$$

each $\phi_i'(x)$ overlaps with its left and right neighbor

↓

tridiagonal



$$\begin{bmatrix} x & x & 0 & 0 & 0 \\ x & x & x & 0 & 0 \\ 0 & x & x & x & 0 \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

$K[0,0] = 1/\Delta x$

$K[1,1] = 2/\Delta x$

$K[i,i] = 2/\Delta x$

$K[0,1] = \int_0^1 \phi_0'\, v_1'\, dx = -1/\Delta x$

$K[i,j] = -1/\Delta x \quad |i-j| = 1$

$$KU = \frac{1}{\Delta x} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \Delta x \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = F$$