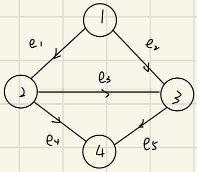


$m=5$ edges

$n=4$ nodes

the incidency matrix $A \in \mathbb{R}^{m\times n}$

$$A = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{c} \begin{array}{cccc} n_1 & n_2 & n_3 & n_4 \end{array} \\ \begin{bmatrix} -1 & 1 & & \\ -1 & & 1 & \\ & -1 & 1 & \\ & -1 & & 1 \\ & & -1 & 1 \end{bmatrix} \end{array}$$

$Au=0$ for $u=\mathbb{1}$

$\operatorname{rank}(A)=3$

$$A^TA = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$A^TA = D - W$

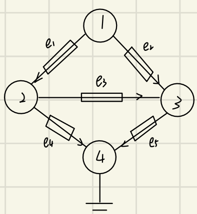
= degree matrix − adjancency matrix

$(A^TA)_{1,4} = 0$: n_1 and n_4 are not connected

$(A^TA)_{2,2} = 3$: degree$(n_2)=3$

$(A^TA)_{i,j} = \langle A[:,i], A[:,j]\rangle = -1 \times$ num of edges between $i\ j$

A^TA is psd



potential at nodes : u

↓

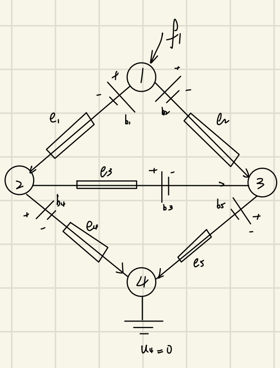
potential difference along edge : $e=-Au$ $\quad$ $e=b-Au$ where b is the source term

↓

currents : $w=Ce$ $\quad$ where $C=1/R$ is the conductance

↓

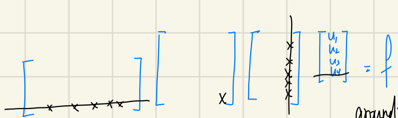
KCL : $A^Tw=f$ $\quad$ where f is external current source



$$\begin{cases} -A^TCAu = f \\ u_4 = 0 \end{cases}$$

A^TCA : conductance matrix (of the system)

ground node 4



$A^T \quad C \quad A$

grounding n_4

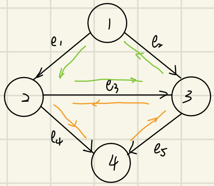
reduced A^TCA is pd

$A^T w = 0$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$\text{rank}(A^T) = 3$

$\dim(\text{nulspace}(A^T)) = 2$



$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

nulspace of A^T gives circulation

$f = A^T w = A^T C e = A^T C(b - Au)$

$A^T C A u = A^T C b - f$ 1 system (Finite element method)

$$\begin{cases} A^T w = f \\ C^{-1} w + Au = b \end{cases}$$

2 systems $\underbrace{\begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix}}_{\text{psd, not pd}} \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$ like KKT system (Mixed method)

$$\begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix} \rightarrow \begin{bmatrix} C^{-1} & A \\ 0 & -A^T C A \end{bmatrix}$$

positive pivots

negative pivots

saddle point