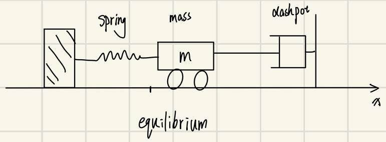
$y'' + Ay' + By = 0$ homogeneous

the general solutions looks like $c_1 y_1 + c_2 y_2$, where y_1 and y_2 are solutions

initial conditions are satisfied by choosing c_1 and c_2



$$mx'' = -kx - cx'$$

$$\downarrow$$

$$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$$

· Solving $y'' + Ay' + By = 0$

find 2 independent solution

Basic method : try e^{rt}

$r^2 \cdot e^{rt} + Ar \cdot e^{rt} + Be^{rt} = 0$ $\qquad r^2 + Ar + B = 0$ characteristic equation

case #1. $r_1 \neq r_2$ are real

$r^2 + Ar + B = 0$ $\qquad y = C_1 e^{r_1 t} + C_2 \cdot e^{r_2 t}$

$A^2 - 4B > 0$

A large

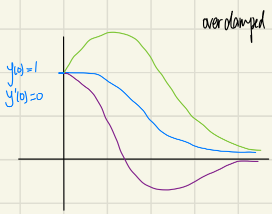
over damped

$$\begin{cases} y'' + 4y' + 3y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

$r^2 + 4r + 3 = 0$

$y = C_1 \cdot e^{-3t} + C_2 \cdot e^{-t}$

$y = -\frac{1}{2} \cdot e^{-3t} + \frac{3}{2} \cdot e^{-t}$



case #2. complex roots $r = a \pm bi$

get solution $y = e^{(a+bi)t}$

$A^2 - 4B < 0$

A small

underdamped

if $u(t) + iv(t)$ is a complex solution to a real differential equation $y'' + Ay' + By = 0$ (A, B are real constants), then u, v are real solutions

$$(u+iv)'' + A(u+iv)' + B(u+iv) = 0$$

$$(u'' + Au' + Bu) + i(v'' + Av' + Bv) = 0$$

$$y = e^{(a+bi)t} = e^{at}\cdot e^{ibt} = e^{at}(\cos bt + i\sin bt)$$

$$\begin{cases} y_1 = e^{at}\cos bt \\ y_2 = e^{at}\cdot\sin bt \end{cases} \Rightarrow y = \underbrace{e^{at}}_{\text{the exp which controls the amplitude}} \underbrace{(C_1\cdot\cos bt + C_2\cdot\sin bt)}_{\text{a purely sinusoidal oscillation}}$$

$$\begin{cases} y'' + 4y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases} \qquad r^2 + 4r + 5 = 0$$

$$r = -2 \pm i$$

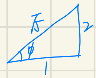
$$e^{(-2+i)t} = e^{-2t}\cdot(\cos t + i\sin t)$$

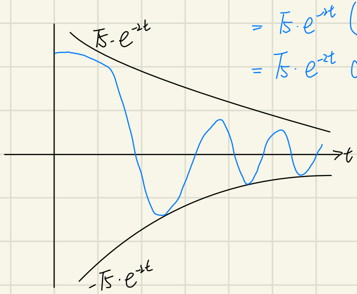
$$\left.\begin{array}{l} y_1 = C_1 e^{-2t}\cos t \\ y_2 = C_2\cdot e^{-2t}\sin t \end{array}\right\} \quad y = e^{-2t}(C_1\cos t + C_2\sin t)$$

$$y = e^{-2t}(\cos t + 2\sin t)$$

$$= \sqrt{5}\cdot e^{-2t}\left(\frac{1}{\sqrt{5}}\cos t + \frac{2}{\sqrt{5}}\sin t\right)$$

$$= \sqrt{5}\cdot e^{-2t}\cos(t - \phi)$$





underdamped

$\boxed{\text{case \#3.}}$ 2 equal real roots $r = -a$

$(r+a)^2 = 0 \quad r = -a$

critically damped
(damping and spring constant are related)

$r^2 + 2ar + a^2 = 0$

$y'' + 2ay' + a^2 y = 0$

have solution $y = e^{-at}$

Knowing the solution y_1 to $y'' + py' + q = 0$

there's another solution $y = y_1 u$

① $y = e^{-at}\cdot u$

② $y' = -a\cdot e^{-at}u + e^{-at}\cdot u'$

③ $y'' = e^{-at}\cdot u'' + -a\cdot e^{-at}\cdot u' + (-a\cdot e^{-at}\cdot u' + a^2 e^{-at} u)$

$\quad = a^2\cdot e^{-at}u - 2a e^{-at}u' + e^{-at}\cdot u''$

$1\times③ + 2a\times② + a^2\times①$

$y'' + 2ay' + a^2 y = 0 + 0 + e^{-at}\cdot u'' := 0$

$u = C_1 t + C_2$

$y_1 = e^{-at}$

$y_2 = e^{-at}(C_1 t + C_2)$ — a whole family of solutions, just pick a simple one

$y = C_1 e^{-at} + C_2\cdot e^{-at}\cdot t$