$$y' + ky = kq(t)$$

$u(t) = e^{\int k\,dt} = e^{kt}$

$e^{kt}\cdot y' + e^{kt}\cdot k\cdot y = q(t)\cdot e^{kt}\, k$

$(y\cdot e^{kt})' = q(t)\cdot e^{kt}\cdot k$

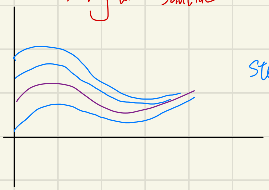
$y\cdot e^{kt} = \int q(t)\cdot e^{kt}\cdot dt + C$

$y(t) = e^{-kt}\cdot \int q(t)\cdot e^{kt}\, dt + c\cdot e^{-kt}$

$$y(t) = e^{-kt}\int q(t)\cdot e^{kt}\, dt + c\cdot e^{-kt}$$

steady-state / long term solution

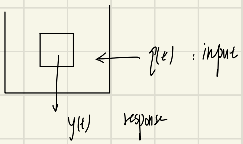
0 as $t \to +\infty$ no matter what C is $(C = y_0)$



Steady-state solution

input : $q(t)$

response : $y(t)$



$y' + ky = k\cdot q(t)$ $(k>0)$:

Superposition of inputs :

$q_1(t) \to y_1(t)$, $q_2(t) \to y_2(t)$ } $q_1(t) + q_2(t) \longrightarrow y_1(t) + y_2(t)$, $cq_1(t) \to cy_1(t)$ linearity

· **Response to trigonometric**

$y' + ky = k\cdot q_e(t)$ where $q_e(t) = \cos wt$

↓ complexify

$\tilde{y}' + k\tilde{y} = k\cdot e^{iwt}$ where $\tilde{y}(t) = y_1(t) + i\cdot y_2(t)$

Find $\tilde{y}$, and take the real part which solves the original ODE

Integrating factor : $u(t) = e^{\int k\,dt} = e^{kt}$

$\tilde{y}'\cdot e^{kt} + k\cdot e^{kt}\,\tilde{y} = k\cdot e^{iwt}\cdot e^{kt}$

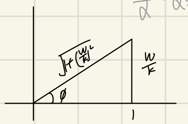
$(\tilde{y}e^{kt})' = k\cdot e^{t(k+iw)}$

$\tilde{y}\, e^{kt} = k\cdot \int e^{t(k+iw)}\, dt = \frac{k}{k+iw} e^{(k+iw)t}$

$\tilde{y} = \frac{k}{k+iw} e^{iwt} = \frac{1}{1+i\cdot\frac{w}{k}}\cdot e^{iwt}$

$\tilde{y}(t) = \frac{1}{1+i\frac{w}{k}}\cdot e^{iwt}$ → go polar; → go cartesian

(1). Go polar



$\frac{1}{\alpha}\cdot\alpha=1 \rightarrow \left|\frac{1}{\alpha}\right| = \frac{1}{|\alpha|}$

$\searrow \text{ang}(\alpha) + \text{ang}(\frac{1}{\alpha}) = 0$

$\left|\frac{1}{a+bi}\right| = \left|\frac{a-bi}{a^2+b^2}\right| = \frac{1}{\sqrt{a^2+b^2}} = \frac{1}{|a+bi|}$

$\text{ang}\left(\frac{1}{1+i\frac{w}{k}}\right) = -\text{ang}\left(1+i\frac{w}{k}\right) = \phi$

$\therefore \frac{1}{1+i\frac{w}{k}} = A\cdot e^{-i\phi} = \frac{1}{\sqrt{1+(\frac{w}{k})^2}}\cdot e^{-i\phi}$

$$\tilde{y}(t) = \frac{1}{1+i\frac{w}{k}}\cdot e^{iwt} = \frac{1}{\sqrt{1+(\frac{w}{k})^2}}\cdot e^{i\phi}\cdot e^{iwt}$$
$$= \frac{1}{\sqrt{1+(\frac{w}{k})^2}}\cdot e^{i(wt-\phi)}$$
$$= \frac{1}{\sqrt{1+(\frac{w}{k})^2}}\cdot\left[\cos(wt-\phi)+i\sin(wt-\phi)\right]$$

$$y(t) = \frac{1}{\sqrt{1+(\frac{w}{k})^2}}\cdot\cos(wt-\phi) \qquad \phi = \arctan(\tfrac{w}{k})$$

phase delay

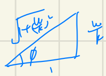
when $k\uparrow$, $\frac{1}{\sqrt{1+(\frac{w}{k})^2}}$ (conductivity up, more turbulence)

$\phi = \arctan\frac{w}{k} \downarrow$

(2). Go Cartesian

$$\tilde{y} = \frac{1}{1+i\frac{w}{k}}\cdot e^{iwt}$$
$$= \frac{1-i\frac{w}{k}}{1+(\frac{w}{k})^2}\left(\cos wt + i\sin wt\right)$$

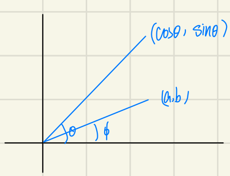
$$y(t) = \frac{1}{1+(\frac{w}{k})^2}\left(\cos wt + \frac{w}{k}\sin wt\right)$$
$$= \frac{1}{1+(\frac{w}{k})^2}\sqrt{1+(\frac{w}{k})^2}\cos(wt-\phi)$$
$$= \frac{1}{\sqrt{1+(\frac{w}{k})^2}}\cdot\cos(wt-\phi)$$



$\cos\phi = \frac{1}{\sqrt{1+(\frac{w}{k})^2}} \qquad \sin\phi = \frac{\frac{w}{k}}{\sqrt{1+(\frac{w}{k})^2}}$

$$a\cos\theta + b\sin\theta = c\cdot\cos(\theta-\phi)$$

18.02 proof



$\langle a,b\rangle\cdot\langle\cos\theta,\sin\theta\rangle$

$= |\langle a,b\rangle|\cdot|\langle\cos\theta,\sin\theta\rangle|\cdot\cos(\theta-\phi)$

$= \sqrt{a^2+b^2}\cdot\cos(\theta-\phi)$

18.03 proof

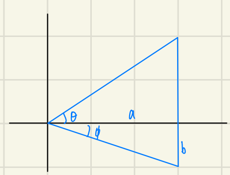
$(a-bi)(\cos\theta+i\sin\theta)$

$= \sqrt{a^2+b^2}\cdot e^{-i\phi}\cdot e^{i\theta}$

$= \sqrt{a^2+b^2}\cdot e^{i(\theta-\phi)}$

$= \sqrt{a^2+b^2}\cdot\left[\cos(\theta-\phi)+i\sin(\theta-\phi)\right]$

take real part



· Recap for Linear ODE

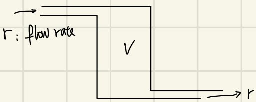
$y' + ky = kq(t)$ — temperature model

$y' + ky = q(t)$

$y' + p(t)y = q(t)$ — Integrating Factor

$k > 0$

eg. mixing



$X(t)$: amount of salt in tank at time t

C_e : concentration of incoming salt

$$\frac{dx}{dt} = (\text{salt in flow}) - (\text{salt out flow})$$

$$= r \cdot C_e - r \cdot \frac{x}{V}$$

$$\frac{dx}{dt} + \frac{r}{V} \cdot x = r \cdot C_e$$

$$\downarrow \quad x/V = C \quad \frac{dx}{dt} = V \cdot \frac{dc}{dt}$$

$$V \cdot \frac{dc}{dt} + r \cdot C = r \cdot C_e$$

$$\frac{dc}{dt} + \frac{r}{V} \cdot C = \frac{r}{V} \cdot C_e$$

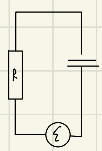
basic parameter $k = \frac{r}{V}$: fractional rate of inflow / outflow

$$\frac{\text{volume / minute}}{\text{volume}} = \text{minute}^{-1}$$

suppose C_e is sinusoidal, how closely does $C(t)$ follows $C_e(t)$

if $k = r/V$ large, follow closed (large conductivity)

eg.



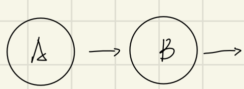
q = charge in capacitator

$dq/dt = i$

$$R \cdot \frac{dq}{dt} + \frac{q}{C} = \varepsilon(t)$$

$$q' + \underbrace{\frac{1}{RC}}_{k} \cdot q = \frac{\varepsilon(t)}{R}$$

eg. chaining decay



$$\frac{dB}{dt} = k_1 A - k_2 B$$

$$= k_1 \cdot A_0 \cdot e^{-k_1 t} - k_2 B \quad \text{radio active decay law}$$

$$B' + k_2 B = k_1 A_0 \cdot e^{-k_1 t}$$

eg. if $k < 0$ $a = -k > 0$

$$y' - ay = q(t)$$

eg. biology $\frac{dP}{dt} = aP$ for population growth

$$y = e^{at} \int q(t) e^{-at} \, dt + \underbrace{C \cdot e^{at}}$$

goes to $+\infty$ or $-\infty$ depending on the initial state