· Substitution

1. change unit
2. make variable dimensionless (no units, eg. gram. second)
3. reduce / simplify the constants.

eg.

$$\frac{dT}{dt} = k(M^4 - T^4)$$

M constant external temperature

T internal temperature

$T_1 = T/M$

$T_1 = T/M$ is unitless

$$M\frac{dT_1}{dt} = kM^4(1 - T_1^4)$$

$$\frac{dT_1}{dt} = kM^4(1-T_1^4) = k_1(1-T_1^4)$$

$k_1 = kM^3$ "lump constants"

· Direct substitution

new var = f(old var)

eg. Bernoulli equation

$y' = p(x)y + q(x)\cdot y^n$ $(n \neq 0)$

$$\frac{y'}{y^n} = p(x)\cdot\frac{1}{y^{n-1}} + q(x)$$

$v := \frac{1}{y^{n-1}} = y^{1-n}$ $v' = (1-n)\cdot y^{-n}\cdot y' = \frac{1-n}{y^n}\cdot y'$

$$\frac{1}{1-n}\cdot v' = p(x)\cdot v + q(x)$$ "linear"

eg. $y' = y/x - y^2$

$$\frac{y'}{y^2} = \frac{1}{x}\cdot\frac{1}{y} - 1$$ $v = \frac{1}{y}$ $v' = -1\cdot\frac{1}{y^2}\cdot y'$

$-v' = \frac{1}{x}\cdot v - 1$

$v' + \frac{1}{x}v = 1$

① Integrating factor: $u(x) = e^{\int p(x)\,dx} = x$

② $xv' + v = x$

③ $(xv)' = x$

④ $xv = \int x\,dx + C = \frac{x^2}{2} + C$ $v = \frac{x}{2} + \frac{C}{x} = \frac{x^2+2C}{2x} = \frac{x^2+C}{2x}$ $y = \frac{1}{v} = \frac{2x}{x^2+C}$

· Inverse substitution

homogeneous ODE : $y' = f(y/x)$ zoom - invariant $y_1 = ay$ $x_1 = ax$

eg $y' = \frac{x^2 y}{x^3+y^3} = \frac{y/x}{1+(y/x)^2}$

lhs: $dy/dx = dy_1/dx_1$

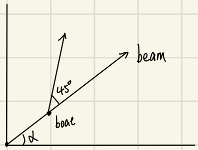
rhs: $f(y/x) = f(y_1/x_1)$

$y' = f(y/x)$ $z := y/x$ $y' = (xz)' = xz' + z$

$xz' + z = f(z)$

$x\frac{dz}{dx} = f(z) - z$ Separate variables

eg. drug boat and beam



the path of boat

know how state changes (direction of boat) ⇒ differential equation

$y' = \tan(\alpha + 45°)$ $\tan\alpha = y/x$

$= \frac{\tan\alpha + \tan 45°}{1 - \tan\alpha \cdot \tan 45°} = \frac{1 + y/x}{1 - y/x}$

$y' = \frac{1+y/x}{1-y/x}$ $z = y/x$ $y' = z + z'x$

$z + z'x = \frac{1+z}{1-z}$

$\frac{x\,dz}{dx} = \frac{1+z}{1-z} - z = \frac{1+z^2}{1-z}$

$\frac{1}{x}dx = \frac{1-z}{1+z^2}dz$

$\ln x = \arctan z - \int \frac{z}{1+z^2}dz = \arctan z - \frac{1}{2}\ln(1+z^2) + C$

$\ln x = \arctan z - \ln\sqrt{1+z^2} + C$

$\ln x = \arctan y/x - \ln\frac{\sqrt{x^2+y^2}}{x} + C$

$= \arctan y/x - \ln\sqrt{x^2+y^2} + \ln x + C$

$\arctan y/x = \ln\sqrt{x^2+y^2} + C$

$\theta = \ln r + C$ $r = C\cdot e^{\theta}$