$$\mathcal{L}f(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$
$$\mathcal{L}g(s) = \int_0^{\infty} e^{-st} g(t)\, dt$$

$$\mathcal{L}f(s) \cdot \mathcal{L}g(s) = \mathcal{L}(f * g)(s)$$

Similar to Fourier Transform : Convolution in time domain is multiplication in frequency domain (22 261)

· Analogy

$$F(x) = \sum_{n=0}^{+\infty} a_n \cdot x^n \quad \rightarrow \quad \underset{x\in(0,1)}{\int_0^{\infty} a(t)\, x^t\, dt} \quad \xrightarrow{x^t = (e^{\ln x})^t} \quad \int_0^{+\infty} a(t)\, e^{-st}\, dt \quad (s>0)$$
$$G(x) = \sum_{n=0}^{\infty} b_n \cdot x^n$$

$$F(x) \cdot G(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} C_n x^n$$
$$C_n = \sum_{k=0}^{n} a_k \cdot b_{n-k} = \sum_{k=0}^{n} a(k) \cdot b(n-k) \quad \rightarrow \quad (f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$$
$$f * g = g * f$$

eg $t^2 * t$

$$\int_0^t u^2 \cdot (t-u)\, du$$
$$= \frac{u^3}{3} t \Big|_0^t - \frac{u^4}{4}\Big|_0^t = \frac{t^4}{12}$$

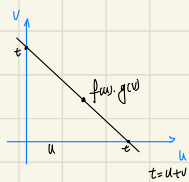
$$\begin{array}{ccccc} t^2 & * & t & = & \frac{t^4}{12} \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} & & \uparrow \mathcal{L}^{-1} \\ \frac{2!}{s^3} & \times & \frac{1}{s^2} & = & \frac{2}{s^5} \end{array}$$

eg. $f * 1$

$$\int_0^t f(u) \cdot 1\, dt = \int_0^t f(u)\, du$$

· Proof

$$\mathcal{L}f(s) \cdot \mathcal{L}g(s) = \int_0^{+\infty} f(u) \cdot e^{-su}\, du \cdot \int_0^{+\infty} g(v) \cdot e^{-sv}\, dv$$
$$= \int_{0_v}^{+\infty} \int_{0_u}^{+\infty} f(u) \cdot g(v)\, e^{-s(u+v)}\, du\, dv$$
$$\overset{t=u+v}{=} \int_{0_t}^{+\infty} \int_{0_u}^{t} f(u)\, g(t-u)\, e^{-st}\, du\, dt$$
$$= \int_{0_t}^{+\infty} e^{-st} \underbrace{\int_{0_u}^{t} f(u)\, g(t-u)\, du}_{(f*g)(t)}\, dt$$

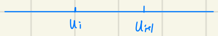


$$u = u \qquad \begin{bmatrix} u \\ t \end{bmatrix}$$
$$t = u + v \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$du\, dv = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} \cdot du\, dt = du\, dt$$

eg. Radioactive waste is dumped with rate $f(t)$

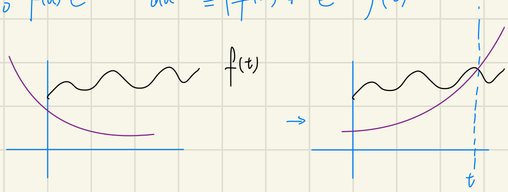
Radio decay: $A_t = A_0 e^{-kt}$



\# dump $[u_i, u_{i+1}] \approx f(u_i) \cdot \Delta u$

By time t, the material dumped in $[u_i, u_{i+1}]$ $\quad f(u_i) \cdot \Delta u \cdot e^{-k(t-u_i)}$

$$\int_0^t f(u)\, e^{-k(t-u)}\, du = \left(f(u) * e^{-ku}\right)(t)$$



when there's no decay $\quad \int_0^t f(u) \cdot 1\, dt = (f * 1)(t)$