$F(S) = \int_0^{+\infty} f(t)\, e^{-St}\, dt$

to make $F(s)$ converge, $f(t)$ can't grow too rapidly so that e^{-st} can pull it down

$f(t)$ of "exponential type"

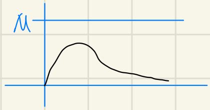
$\exists\ C, k$ st. $|f(t)| \leq C \cdot e^{kt}$ $\forall\ t \geq 0$

eg. $\sin t$ $|\sin t| \leq 1$ $|\sin t| \leq 1 \cdot e^{0t}$

eg t^n $|t^n| \leq M \cdot e^t$ for some M

$\frac{t^n}{e^t}(0) = 0$

$\lim_{t \to \infty} \frac{t^n}{e^t} = \lim_{t \to \infty} \frac{n \cdot t^{n-1}}{e^t} \cdots = \lim_{t \to \infty} \frac{n!}{e^t} = 0$



Start at 0
end at 0
continuous

} has a maximum

eg $1/t$

$\int_0^{+\infty} 1/t \cdot e^{-st}\, dt$

when $t \approx 0$ $e^{-ts} \approx 1$ $\int_0^{\varepsilon} 1/t \cdot 1\, dt$ does not converge

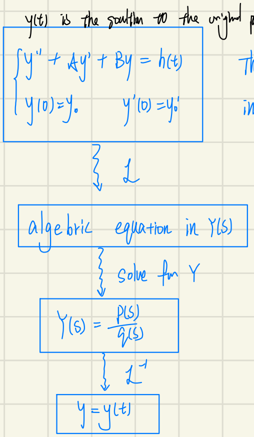
$1/t$ is not of exponential type

$\therefore$ $1/t$ does not have a Laplace transform (improper start)

eg. e^{t^2}

$\exists\ t$ st. $e^{t^2} > e^{kt}$ no matter how big k is (grow too rapidly)

- Solve ODE with Laplace transform



y(t) is the solution to the original problem

$\begin{cases} y'' + Ay' + By = h(t) \\ y(0) = y_0 \quad y'(0) = y_0' \end{cases}$

$\mathcal{L}$

algebric equation in $Y(s)$

solve for Y

$Y(s) = \frac{p(s)}{q(s)}$

$\mathcal{L}^{-1}$

$y = y(t)$

The Laplace transform must have an initial value problem, assume an initial condition if not

· Laplace Transform of derivative

$$(\mathcal{L}f')(s) = \int_0^{+\infty} f'(t)\, e^{-st}\, dt$$

$$= \int_0^{+\infty} e^{-st}\, d f(t) = e^{-st} f(t) \Big|_0^{+\infty} - \int_0^{+\infty} f(t)\, d e^{-st}$$

$$= e^{-st} f(t) \Big|_0^{+\infty} + s\int_0^{+\infty} f(t)\, e^{-st}\, dt$$

when $f(t)$ is exponential type $\lim_{t\to+\infty} \frac{f(t)}{e^{st}} = 0$ $(s>0)$

$$= -f(0) + s\int_0^{+\infty} f(t)\, e^{-st}\, dt$$

$$(\mathcal{L}f)(s) = -f(0) + s\cdot(\mathcal{L}f)(s)$$

$$(\mathcal{L}f'')(s) = -f'(0) + s\cdot(\mathcal{L}f')(s)$$

$$= -f'(0) + s\left[-f(0) + s\cdot(\mathcal{L}f)(s)\right]$$

$$(\mathcal{L}f')(s) = s^2 \mathcal{L}f(s) - sf(0) - f'(0)$$

eg. $y'' - y = e^{-t}$

① Solve for operator

$y'' - y = 0$ $y = e^{rt}$

$r^2 - 1 = 0$ $y = e^{t}, e^{-t}$ $y_c = C_1\cdot e^{t} + C_2\cdot e^{-t}$

$y'' - y = e^{-t}$ $p(D) = D^2 - 1$

$p(D)y = e^{-t}$

$y = \frac{e^{-t}}{p(-1)} = \frac{e^{-t}}{0}$ (nope)

$\downarrow$

$\frac{t\cdot e^{-t}}{p'(-1)} = \frac{-t\cdot e^{-t}}{2}$ $y_p = -\frac{t\cdot e^{-t}}{2} + C_1 e^{t} + C_2\cdot e^{-t}$

② Solve by Laplace transform

$$\begin{cases} y'' - y = e^{-t} \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

$(\mathcal{L}y'')(s) = s^2(\mathcal{L}y)(s) - sy(0) - y'(0) = s^2(\mathcal{L}y)(s) - s$

$\mathcal{L}(e^{-t})(s) = \frac{1}{s+1}$

$$s^2\mathcal{L}y - s + \mathcal{L}y = \frac{1}{s+1}$$

$$\begin{array}{llll} (\mathcal{L}y)(s) = \frac{s^2+s+1}{(s+1)^2(s-1)} & = \frac{-1/2}{(s+1)^2} & + \frac{1/4}{s+1} & + \frac{3/4}{s-1} \\ \Big\downarrow \mathcal{L}^{-1} & \Big\downarrow \mathcal{L}^{-1} & \Big\downarrow \mathcal{L}^{-1} & \Big\downarrow \mathcal{L}^{-1} \\ y(t) = & -1/2\ t\cdot e^{-t} & + \frac{1}{4} e^{-t} & + \frac{3}{4} e^{t} \end{array}$$

$$\left.\begin{array}{ll} e^{at} f(t) \longrightarrow \mathcal{L}(s-a) & \\ \mathcal{L}(t^n) \to \frac{n!}{s^{n+1}} & \mathcal{L}(t) \to \frac{1}{s^2} \end{array}\right] \quad t\cdot e^{-t} \longrightarrow t\cdot e^{-t}$$