power series $\sum_{n\geq 0}^{\infty} a_n x^n = A(x)$

$\downarrow$

$\sum_{n=0}^{\infty} a(n) x^n = A(x)$ take the discrete function $a(n)$ and associate $a(n)$ to $A(x)$

eg. $a(n) = 1$ $A(x) = 1 + x + x^2 \cdots$

eg. $a(n) = 1/n!$ $A(x) = 1 x' + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 \cdots = e^x$

continuous analog

$t \in [0, +\infty]$ $n = 0, 1, 2 \cdots$

$\int_0^{+\infty} a(t)\, x^t\, dt = A(x)$ $\sum_{n=0}^{+\infty} a(n)\, x^n = A(x)$

$\downarrow$ $x^t = (e^{\ln x})^t$

$x < 1$ to converge ; $x > 0$ to behave properly (what's $-1^{\frac{1}{2}}$)

$\downarrow$

$x \in (0,1)$ $\ln x \in (-\infty, 0)$ $s = -\ln x = (0, +\infty)$

$\int_0^{+\infty} f(t)\, e^{-st}\, dt = F(s)$ Laplace transform $f(t) \rightsquigarrow F(s)$

$f(t) \to \boxed{\text{Transform}} \to F(s)$ $f(t) \to \boxed{\text{operator}} \to g(t)$

Laplace transform is linear

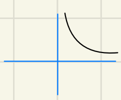
$\mathcal{L}(f+g) = \int_0^{+\infty} (f(t) + g(t))\, e^{-st}\, dt = \mathcal{L}(f) + \mathcal{L}(g)$

$\mathcal{L}(cf) = \int_0^{+\infty} c\, f(t)\, e^{-st}\, dt = c \cdot \mathcal{L}(f)$

eg $f(t) = 1$

$\mathcal{L}(f) = \int_0^{+\infty} f(t) \cdot e^{-st}\, dt = \lim_{R \to +\infty} \int_0^R e^{-st}\, dt$

$= \lim_{R \to +\infty} \left(-\frac{1}{s} e^{-st} \Big|_0^R \right) = = \begin{cases} 1/s & \text{if } s > 0 \\ \text{meaningless} & \text{if } s < 0 \end{cases}$



eg. $e^{at} f(t)$ $\qquad e^{at} f(t) \to F(s-a)$ $\qquad \left\{ \begin{array}{l} f(t)=1 \to F(s)=1/s \\ e^{at} f(t) \to F(s-a) \end{array} \right\} e^{at} \cdot 1 = \frac{1}{s-a}$

$$\int_0^{+\infty} e^{at} f(t) e^{-st} dt = \int_0^{+\infty} f(t) e^{-(s-a)t} dt = F(s-a) \quad \text{when } s>a$$

exponential-shift law idea

eg. $\cos(at)$ $\qquad \cos(at) \rightsquigarrow s/s^2+a^2$

$\sin(at) \rightsquigarrow a/s^2+a^2$

$$\cos(at) = \frac{e^{iat}+e^{-iat}}{2}$$

$$\mathcal{L}(\cos at) = \frac{1}{2}\left[\mathcal{L}(e^{iat}) + \mathcal{L}(e^{-iat})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right]$$

$\frac{1}{s-ia} + \frac{1}{s+ia}$ is real (when change i to $-i$, it doesn't change)

$$= s/(s^2+a^2)$$

Inverse Laplace transform

$$\frac{1}{s(s+3)} = \frac{1/3}{s} + \frac{-1/3}{s+3}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s+3)}\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{3}\mathcal{L}^{-1}(s+3)$$

$$= \frac{1}{3}\cdot 1 - \frac{1}{3} e^{-3t}$$

eg. t^n

$$\mathcal{L}(t^n) = \int_0^{+\infty} t^n \cdot e^{-st} dt = \int_0^{+\infty} t^n \, d\frac{e^{-st}}{-s}$$

$$= t^n \cdot \frac{e^{-st}}{-s}\Big|_0^{+\infty} - \int_0^{+\infty} -\frac{1}{s} e^{-st} dt^n$$

$$= \lim_{t\to\infty}\left[t^n e^{-st} \cdot \frac{-1}{s}\right] + \frac{1}{ns}\int_0^{+\infty} e^{-st} t^{n-1} dt$$

$(s>0)$

$$\mathcal{L}(t^n) = 0 + \frac{1}{ns}\mathcal{L}(t^{n-1})$$

$$\mathcal{L}(t^n) = \frac{n!}{s^n} \cdot \mathcal{L}(t^0) = \frac{n!}{s^{n+1}}$$