$f(t)$ is periodic, with a period 2π, can always be represented as (infinite) sums of sin and cos

$$f(t) = C_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

$u(t)$ $v(t)$ has a period 2π

$u(t)$ and $v(t)$ are orthogonal if $\int_{-\pi}^{\pi} u(t)\cdot v(t)\, dt = 0$

any two distinct member of $\{\cos mt, \sin mt\}$ are orthogonal

$$\int_{-\pi}^{\pi} \sin^2 nt \, dt = \int_{-\pi}^{\pi} \cos^2 nt \, dt = \pi$$

· Identifying orthogonality by ODE

$u'' + n^2 u = 0 \quad \rightarrow \quad u'' = -n^2 u$

let $u_n, v_m \in \{\cos kt, \sin kt\}$, both satisfies $u'' = -n^2 u$ and $m \neq n$

$$\int_{-\pi}^{\pi} u_n'' v_m \, dt = \underbrace{u_n' v_m \Big|_{-\pi}^{\pi}}_{0} - \int_{-\pi}^{\pi} u_n' v_m' \, dt = -\int_{-\pi}^{\pi} u_n' v_m' \, dt$$

$$\underbrace{-n^2 \int_{-\pi}^{\pi} u_n v_m \, dt}_{\text{not symmetric, } -n^2 \text{ favors } u} = \underbrace{-\int_{-\pi}^{\pi} u_n' v_m' \, dt}_{\text{symmetric in } u \text{ and } v}$$

the only way this can happen is $\int_{-\pi}^{\pi} u_n \cdot v_m \, dt = 0$ $(m \neq n)$

$$\begin{cases} -\int_{-\pi}^{\pi} u_n' v_m' \, dt = -n^2 \int_{-\pi}^{\pi} u_n v_m \, dt \\ -\int_{-\pi}^{\pi} v_m' u_n' \, dt = -m^2 \int_{-\pi}^{\pi} v_m u_n \, dt \end{cases}$$ should be equality

· a_n & b_n

$$f(t) = a_0 \cos 0t + \sum_{m=1}^{\infty} a_m \cos mt + b_m \sin mt$$

$$f(t)\cdot \cos nt = a_0 \cos 0t \cdot \cos nt + \sum_{m=1}^{\infty} a_m \cos mt \cos nt + b_m \sin mt \cos nt$$

$$\int_{-\pi}^{\pi} f(t) \cos nt \, dt = \int_{-\pi}^{\pi} a_0 \cos 0t \cos nt \, dt + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mt \cos nt \, dt + b_m \int_{-\pi}^{\pi} \sin mt \cos nt \, dt$$

$$= a_n \int_{-\pi}^{\pi} \cos nt \cos nt \, dt = a_n \pi$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$$

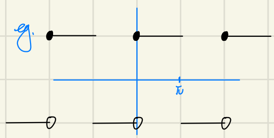
$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos mt + b_m \sin mt$$

$\Downarrow$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$$

eg.



$a_n = 0$ asymmetry

$$b_n = -\int_{-\pi}^{0} \sin nt \, dt + \int_{0}^{\pi} \sin nt \, dt$$

$$= \frac{1-\cos n\pi}{n} + \frac{1-\cos n\pi}{n}$$

$$= \frac{2}{n}(1-\cos n\pi)$$

$$= \frac{2}{n}(1-(-1)^n)$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{n} & n \text{ odd} \end{cases}$$