$p(D)\,y = e^{\alpha t}$

$$y_p = \begin{cases} \dfrac{e^{\alpha t}}{p(\alpha)} & p(\alpha) \neq 0 \\ \dfrac{t \cdot e^{\alpha t}}{p'(\alpha)} & p(\alpha)=0 \quad p'(\alpha) \neq 0 \\ \dfrac{t^2 \cdot e^{\alpha t}}{p''(\alpha)} & p'(\alpha)=0 \quad p''(\alpha) \neq 0 \end{cases}$$

· Resonance

(push a swing)

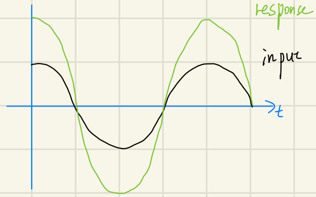
$$y'' + \underbrace{\omega_0^2}_{\text{natural frequency (spring, pendulum, ...)}} y = \overset{\text{input / forcing term}}{\cos \omega_1 t}$$

$(D^2 + \omega_0^2)\,y = \cos \omega_1 t$

$(D^2 + \omega_0^2)\,\tilde{y} = e^{i\omega_1 t}$

$$\tilde{y} = \frac{e^{i\omega_1 t}}{(i\omega_1)^2 + \omega_0^2} = \frac{e^{i\omega_1 t}}{\omega_0^2 - \omega_1^2}$$

$$y = \underbrace{\frac{\cos \omega_1 t}{\omega_0^2 - \omega_1^2}}_{\text{Response}} \quad \begin{matrix} \} \text{ frequency of response} \\ \} \text{ amplitude} \end{matrix}$$



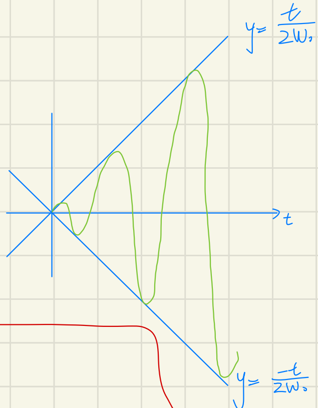
if $\omega_1 = \omega_0$

$y'' + \omega_0^2 y = \cos \omega_0 t$

$(D^2 + \omega_0^2)\,\tilde{y} = e^{i\omega_0 t} \qquad p(i\omega_0) = 0$

$$\tilde{y} = \frac{t \cdot e^{i\omega_0 t}}{p'(i\omega_0)} = \frac{t \cdot e^{i\omega_0 t}}{2 \cdot i\omega_0}$$

$$y = \frac{t \sin \omega_0 t}{2\omega_0}$$



How this change happen

complementary solution

$(D^2 + \omega_0^2)\,y = 0$

$y_c = a \cdot \cos \omega_0 t + b \sin \omega_0 t$

$$y = \underbrace{\frac{\cos \omega_1 t}{\omega_0^2 - \omega_1^2}}_{y_p} - \underbrace{\frac{\cos \omega_0 t}{\omega_0^2 - \omega_1^2}}_{y_c}$$

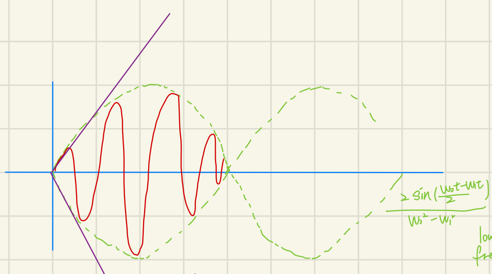
$$\lim_{\omega_1 \to \omega_0} \frac{\cos \omega_1 t - \cos \omega_0 t}{\omega_0^2 - \omega_1^2}$$

$$= \lim_{\omega_1 \to \omega_0} \frac{-t \sin \omega_1 t}{-2\omega_1} = \frac{t \sin \omega_1 t}{2\omega_1}$$

$$\frac{\cos \omega_1 t - \cos \omega_0 t}{\omega_0^2 - \omega_1^2}$$

$$= \underbrace{\frac{2 \sin\left(\frac{\omega_0 t - \omega_1 t}{2}\right)}{\omega_0^2 - \omega_1^2}}_{\text{amplitude}} \underbrace{\sin\left(\frac{\omega_0 t + \omega_1 t}{2}\right)}_{\approx \sin \omega_1 t \quad \text{oscillation}}$$

low frequency $\left\{ \sin\left(\frac{\omega_0 - \omega_1}{2} \cdot t\right) \right.$

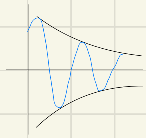


the period get large as $\omega_1 \to \omega_0$

damped case

$$mx'' + cx' + kx = f_1(t)$$

$$x'' + 2px' + \underbrace{w_0^2}_{\text{natural undamped frequency}} x = f_2(t) \qquad w_1 = \sqrt{w_0^2 - p^2} \text{ is the pseudo frequency}$$



(lec 10)

damped case with driving term

$$y'' + 2py' + w_0^2 y = \cos wt$$

what w in input gives maximum amplitude in output

$$w^* = \sqrt{w_0^2 - 2p^2}$$