$y'' + Ay' + By = f(x)$

important $f(x)$ $\left\{\begin{array}{ll} e^{ax} & (\text{often } a<0) \\ \sin wx & \cos wx \\ e^{ax}\sin wx & e^{ax}\cos wx \end{array}\right\} e^{(a+iw)x}$ $\Big]$ $e^{\alpha x}$ where α is a complex number

$\downarrow$

$(D^2 + AD + B)y = P(D)y = f(x)$

$P(D)e^{\alpha x} = P(\alpha)\cdot e^{\alpha x}$

$$(D^2 + AD + B)e^{\alpha x} = D^2 e^{\alpha x} + ADe^{\alpha x} + Be^{\alpha x}$$
$$= \alpha^2 e^{\alpha x} + A\cdot\alpha\cdot e^{\alpha x} + Be^{\alpha x}$$
$$= (\alpha^2 + A\alpha + B)e^{\alpha x}$$
$$P(D)e^{\alpha x} = P(\alpha)e^{\alpha x}$$

· Exponential – Input theorem

for $(D^2 + AD + B)y = e^{\alpha x}$ $\qquad y_p = e^{\alpha x}/p(\alpha)$ when $p(\alpha) \neq 0$

$P(D)\cdot e^{\alpha x}/p(\alpha) = p(\alpha)\cdot e^{\alpha x}/p(\alpha) = e^{\alpha x}$ $\qquad$ $P(D)$ and $p(\alpha)$ "cancels" each other

eg. $y'' - y' + 2y = 10\cdot e^{-x}\cdot \sin x$

$\downarrow$ imaginary part of $e^{(-1+i)x} = e^{-x}(\cos x + i\sin x)$

$$(D^2 - D + 2)\tilde{y}_p = 10e^{(-1+i)x} = 10e^{\alpha x}$$
$$\tilde{y}_p = 10\cdot e^{\alpha x}/p(\alpha) = e^{(-1+i)x}/[(-1+i)^2 - (-1+i) + 2]$$
$$= 10.\ e^{(-1+i)x}/(3-3i) = \frac{10}{3}\cdot\frac{1+i}{2}\cdot e^{-x}(\cos x + i\sin x)$$
$$y_p = \frac{5}{3}\cdot e^{-x}(\cos x + \sin x) \qquad \text{(take the imaginary part)}$$

· Exponential shift rule

$P(D)\cdot e^{ax}\cdot u(x) = e^{ax}\,P(D+a)\cdot u(x)$ $\qquad$ where a is a complex number

take $p(D) = D$

$$D\cdot e^{ax}\cdot u(x) = e^{ax}u'(x) + a\cdot e^{ax}\cdot u(x) = e^{ax}(D+a)u(x)$$

take $P(D) = D^2$

$$D^2 e^{ax}u(x) = D[De^{ax}u(x)] = D[e^{ax}(D+a)u(x)]$$
$$= e^{ax}(D+a)(D+a)u(x) = e^{ax}(D+a)^2 u(x)$$

prove $P = D^n$ by mathematical induction

true for $P(D) = D$ and $P(D) = D^2$, then true for $P(D) = aD^2 + bD$ by linearity

· $(D^2 + AD + B)y = e^{ax}$ (a can be complex) $p(a)=0$

1° $\begin{cases} p(a)=0 \\ p'(a)\neq 0 \end{cases}$ a is a simple root of $p(a)=0$

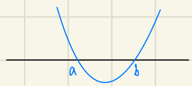
$$y_p = \frac{x\cdot e^{ax}}{p'(a)}$$

2° $\begin{cases} p(a)=0 \\ p'(a)=0 \end{cases}$ a is a double root of $p(a)=a$

$$y_p = \frac{x^2\cdot e^{ax}}{p''(a)}$$

proof 1°

a is a simple root $\begin{cases} p(a)=0 \\ p'(a)\neq 0 \end{cases}$



$p(D) = (D-a)(D-b)$

$p'(D) = (D-a)+(D-b)$

$p'(a) = a-a + a-b = a-b \neq 0$

$$p(D)\frac{x\cdot e^{ax}}{p'(a)} = \frac{p(D)\cdot e^{ax}\cdot x}{p'(a)}$$

$$= \frac{e^{ax}\, p(D+a)x}{p'(a)}$$ by exponential-shift rule

$$= \frac{e^{ax}\,(D+a-b)\cdot Dx}{p'(a)}$$

$$= \frac{e^{ax}(a-b)}{p'(a)} = e^{ax}$$

eg: $y'' - 3y' + 2y = e^x$

$p(D)y = (D^2 - 3D + 2)y = e^x = e^{1x}$

$p(1) = 0 \quad p'(1)\neq 0$

$y_p = x\cdot e^x / p'(1) = -x\cdot e^x$