$y'' + p(x)y' + q(x)y = \underbrace{f(x)}_{\text{input}}$ $\quad$ y: output / response

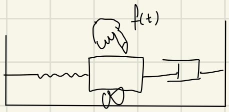
$y'' + p(x)y' + q(x)y = 0$ $\quad$ associated homogeneous equation / reduced equation

$y = c_1 y_1 + c_2 y_2$ $\quad$ general solution / complementary solution $\quad$ (kind of like null space)

eg spring – mass – dashpot

$mx'' + bx' + kx = f(t)$

$\underbrace{mx''}_{\text{force}} = -kx - bx' + \underbrace{f(t)}_{\text{external force}}$



Theorem.

$Ly = f(x)$ $\quad$ $L = D^2 + pD + q$

the solution has the form $y = y_p + y_c = y_p + c_1y_1 + c_2y_2$

$y_p \downarrow$ one particular solution s.t. $Ly = f(x)$

$y_c \downarrow$ general solution s.t. $Ly = 0$

① All $y_p + c_1y_1 + c_2y_2$ are solutions

$L(y_p + c_1y_1 + c_2y_2) = Ly_p + c_1 L(y_1) + c_2 \cdot L(y_2) = f(x)$

② No other solutions

if $u(x)$ is a solution $\left.\begin{array}{l} Lu = f(x) \\ Ly_p = f(x) \end{array}\right\}$ $L(u - y_p) = 0$

$\therefore u - y_p = c_1y_1 + c_2y_2$

$u = c_1y_1 + c_2y_2 + y_p$

· Linear 1st order ODE

$y' + ky = q(t)$

$y = \underbrace{e^{-kt} \int e^{kt} \cdot q(t) \cdot dt}_{y_p} + \underbrace{c \cdot e^{-kt}}_{y_c}$ $\quad$ $(y_c' + k \cdot y_c = 0)$

$u(x) = e^{kt}$

$e^{kt} y' + k \cdot e^{t} = q(t) \cdot e^{kt}$

$(e^{kt} \cdot y)' = q(t) \cdot e^{kt}$

$y = e^{-kt} \cdot \int e^{kt} \cdot q(t)\, dt + c \cdot e^{-kt}$

$k > 0$: $y = y_p$ (steady state) $+ y_c$ (transient)

$k < 0$: above is meaningless

- Linear 2nd order ODE

$$y = \underbrace{y_p}_{\text{Steady state}} + \underbrace{c_1 y_1 + c_2 y_2}_{\text{contains the initial condition}}$$

under what conditions $c_1 y_1 + c_2 y_2 \to 0$ as $t \to +\infty$

characteristic roots	solutions	Stability condition
real $r_1 \neq r_2$	$c_1 \cdot e^{r_1 t} + c_2 \cdot e^{r_2 t}$	$r_1 < 0$ and $r_2 < 0$
real $r_1 = r_2$	$(c_1 + c_2 t) e^{rt}$	$r < 0$
complex $a \pm bi$	$e^{at} (c_1 \cos bt + c_2 \sin bt)$	$a < 0$

The ODE is $\boxed{\text{stable}}$ if all characteristic roots have $\boxed{\text{negative real part}}$