$y'' + p(x)y' + q(x)y = 0$

Standard method: find 2 independent solutions y_1 y_2

then all solutions $y = C_1 y_1 + C_2 y_2$

Q_1: why all $C_1y_1 + C_2y_2$ are solutions?

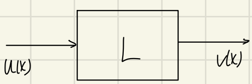
Q_2. why $C_1y_1 + C_2y_2$ are all solutions?

· Superposition principle all $C_1y_1 + C_2y_2$ are solutions

if y_1 and y_2 are solutions to a linear homogeneous ODE (can be higher order)

then $C_1y_1 + C_2y_2$ is a solution

$L = D^2 + pD + q$



$y'' + p(x)y' + q(x)y = 0$

$\downarrow$

$D^2y + pDy + qy = 0$

$\downarrow$

$(D^2 + pD + q)y = 0 \quad \rightarrow \quad Ly = 0$

not multiplying D times y,

apply D to y

Solving differential equations: what's the input so that the outcome is 0

$L = D^2 + pD + q$ is a linear operator $\begin{cases} L(u_1 + u_2) = L(u_1) + L(u_2) \\ L(c \cdot u) = c \cdot L(u) \end{cases}$

eg. D is Linear $D(u_1 + u_2) = D(u_1) + D(u_2)$ $\quad D(c \cdot u) = c \cdot D(u)$

· $C_1y_1 + C_2y_2$ are all solutions

Solving the initial value problem (fit initial values)

$\{C_1y_1 + C_2y_2\}$ is enough to satisfy any initial value $\begin{cases} y(x_0) = a \\ y'(x_0) = b \end{cases}$

$\begin{cases} y = C_1y_1 + C_2y_2 \\ y' = C_1y_1' + C_2y_2' \end{cases} \rightarrow \begin{cases} a = C_1y_1(x_0) + C_2y_2(x_0) \\ b = C_1y_1'(x_0) + C_2y_2'(x_0) \end{cases}$ $\quad \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$ must be invertible $\quad$ non solvable if $y_2 = c \cdot y_1$

wronskian: $w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_2 \cdot y_1' + y_1 \cdot y_2'$ is a function of x

if $y_2 = cy_1$ $\quad w(y_1, y_2) = 0$ for all x

Theorem:

if y_1 y_1 are solutions to linear homogeneous 2nd-order ODE

then Wronskian $w(y_1, y_2)$ either $w(y_1, y_2) = 0$ for all x

or $w(y_1, y_2) \neq 0$ for all x

$\{c_1y_1 + c_2y_2\} = \{a_1u_1 + a_2y_2\}$ where u_1 and u_2 are any other pair of independent solutions

$u_1 = b_1y_1 + b_2y_2$

$u_2 = b_3y_1 + b_4y_2$

(find normalized solutions (at x_0))

· Finding normalized solutions

$y = a_1y_1 + a_2y_2 = c_1y_{n1} + c_2y_{n2}$ where y_{n1} and y_{n2} are normalized solutions

y_{n1}: $y_{n1}(x_0) = 1$, $y_{n1}'(x_0) = 0$

y_{n2}: $y_{n2}(x_0) = 0$, $y_{n2}'(x_0) = 1$

eg. $y'' + y = 0$

$y = e^{rt}$ $\quad r^2 + 1 = 0 \quad y = \pm i$

$e^{rt} = e^{\pm it} = \cos t \pm \sin t$

$y = c_1 \cos t + c_2 \sin t$

$y_1 = \cos t$ $\quad \begin{cases} y_1(0) = 1 \\ y_1'(0) = 0 \end{cases}$ $\quad \begin{cases} y_2(0) = 0 \\ y_2'(0) = 1 \end{cases}$

$y_2 = \sin t$

eg $y'' - y = 0$

$y_1 = e^t \quad y_2 = e^{-t}$ $\qquad y = c_1e^t + c_2e^{-t}$

$y' = c_1e^t - c_2e^{-t}$

$y_{n1}: \begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = 0 \end{cases}$ $\quad \begin{cases} c_1 = \frac{1}{2} \\ c_2 = \frac{1}{2} \end{cases}$ $\qquad y_{n2}: \begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 1 \end{cases}$ $\quad \begin{cases} c_1 = \frac{1}{2} \\ c_2 = -\frac{1}{2} \end{cases}$

$y_{n1} = \frac{e^x + e^{-x}}{2} = \cosh(x)$ $\qquad y_{n2} = \frac{e^x - e^{-x}}{2} = \sinh(x)$

$\begin{cases} y(0) = a \\ y'(0) = b \end{cases} \rightarrow y = a \cdot y_{n1} + b \cdot y_{n2}$

· $c_1y_1 + c_2y_2$ are all solutions

$y'' + py' + py = 0$ where p and q are continuous,
there's one and only one solution that satisfies the initial values
] the existence and uniqueness theorem

and $c_1y_1 + c_2y_2$ can satisfy all initial conditions