$y'' + by' + ky = 0 \xrightarrow{r^2+br+k=0}$ complex root

$y = e^{rt}$

$r = a \pm bi$

$y = e^{(a\pm bi)t}$ $\xrightarrow{\text{real}} e^{at}\cos bt$, $\xrightarrow{\text{img}} e^{at}\sin bt$ $\Big\}$ $y = e^{at}(C_1\cos bt + C_2 \sin bt)$

· **Another approach**

$y = C_1 \cdot e^{(a+bi)} + C_2 \cdot e^{(a-bi)}$

to get real a solution, C_1 and C_2 must be complex numbers

want $u+iv$ to be real $\longrightarrow v=0$ (multiply out)

change i to $-i$, $u+iv$ should stay the same

$C_1 \cdot e^{(a+bi)t} + C_2 \cdot e^{(a-bi)t} \xrightarrow{i \to -i} \overline{C_1} \cdot e^{(a-bi)t} + \overline{C_2} \cdot e^{(a+bi)t}$

want these to be the same.

$\left.\begin{array}{l} \overline{C_2} = C_1 \\ C_2 = \overline{C_1} \end{array}\right\} \to \overline{C_2} = C_1$

$C_1 = c + di$

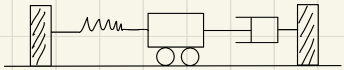
$$\begin{aligned} y &= (c+di)e^{(a+bi)t} + (c-di)e^{(a-bi)t} \\ &= e^{at}\left[(c+di)e^{ibt} + (c-di)e^{-ibt}\right] \\ &= e^{at}\left[c\cdot(e^{ibt} + e^{-ibt}) + id(e^{ibt} - e^{-ibt})\right] \\ &= e^{at}\left[2c\cdot\cos bt + id\cdot 2i\cdot \sin bt\right] \\ &= e^{at}\left[2c\cos bt - 2d \sin bt\right] \end{aligned}$$

$e^{it} = \cos t + i \sin a$

$\cos t = (e^{it} + e^{-it})/2$

$\sin t = (e^{it} - e^{-it})/2i$

· **Oscillation**



$mx'' + cx' + kx = 0$

$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$

$y'' + 2py' + w_0^2 y = 0$

$y = e^{rt}$ $\quad r^2 + 2pr + w_0^2 = 0$

$$r = \frac{2p \pm \sqrt{4p^2 - 4w_0^2}}{2} = p \pm \sqrt{p^2 - w_0^2}$$

1° $p=0$ $(c=0)$ undamped (no dashpot)

$y'' + w_0^2 y = 0$ $\quad r^2 + w_0^2 r = 0$

w_0: circular frequency

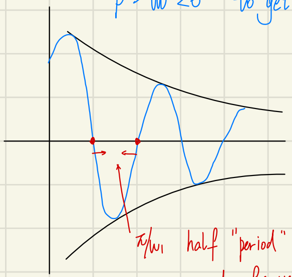
$r = \pm i w_0$

$e^{\pm i w_0 t} = \cos w_0 t \pm i \sin w_0 t$

$y = C_1 \cdot \cos w_0 t + C_2 \cdot \sin w_0 t = A \cdot \cos(wt - \phi)$

2° damped

$p^2 - \omega_0^2 < 0$ to get complex roots $\quad p < \omega_0$



$r^2 + 2pr + \omega_0^2 = 0$

$r = p \pm \sqrt{p^2 - \omega_0^2} = p \pm i\sqrt{\omega_0^2 - p^2}$

$e^{rt} = e^{(p + i\sqrt{\omega_0^2 - p^2})t} = e^{pt} \cdot (\cos\sqrt{\omega_0^2 - p^2}\,t + i\sin\sqrt{\omega_0^2 - p^2}\,t)$

$y = e^{pt} \cdot \left(C_1 \cos\sqrt{\omega_0^2 - p^2}\,t + C_2 \sin\sqrt{\omega_0^2 - p^2}\,t\right)$

$\quad = e^{pt} \cdot A \cdot \sin(\sqrt{\omega_0^2 - p^2}\,t + \phi)$

π/ω_1 half "period"

ω_1 : pseudo frequency $\sqrt{\omega_0^2 - p^2}$

if the damping goes up $\quad c\uparrow$, $p\uparrow$ $\sqrt{\omega_0^2 - p^2}\downarrow$

$p = c/m$

$\omega_1 = \sqrt{\omega_0^2 - p^2}$ } depends only on ODE

$\phi = \arccos\left(\frac{C_2}{\sqrt{C_1^2 + C_2^2}}\right)$

$A = \sqrt{C_1^2 + C_2^2}$ } depends only on initial condition